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Global existence of weak solutions to the three-dimensional Euler equations with helical symmetry ☆

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Abstract

In this paper, we mainly investigate the weak solutions of the three-dimensional incompressible Euler equations with helical symmetry in the whole space when the helical swirl vanishes. Specifically, we establish the global existence of weak solutions when the initial vorticity lies in $L^1 \cap L^p$ with $p > 1$. Our result extends the previous work [2], where the initial vorticity is compactly supported and belongs to L^p with $p > 4/3$. The key ingredient in this paper involves the explicit analysis of Biot–Savart law with helical symmetry in domain $\mathbb{R}^2 \times [-\pi, \pi]$ via the theories of singular integral operators and second order elliptic equations.

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Keywords: Incompressible Euler equations; Weak solutions; Helical symmetry; Biot–Savart law

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1. Introduction

The three-dimensional unsteady incompressible Euler equations read as

$$\begin{cases} \partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla p = 0, \\ \operatorname{div} \mathbf{u} = 0, \end{cases} \tag{1.1}$$

for $(t, x) \in \mathbb{R}^+ \times \mathbb{R}^3$, where the unknowns $\mathbf{u} = (u_1, u_2, u_3)$ and $p = p(t, x)$ represent the velocity fields and the pressure of the fluid, respectively. The corresponding vorticity, $\boldsymbol{\omega} = \operatorname{curl} \mathbf{u}$ satisfies the equation

$$\partial_t \boldsymbol{\omega} + (\mathbf{u} \cdot \nabla)\boldsymbol{\omega} = (\boldsymbol{\omega} \cdot \nabla)\mathbf{u}. \tag{1.2}$$

As is well known, the global existence of weak solutions to the three-dimensional Euler equations remains an open problem due to the strong nonlinearity of the vortex stretching term $(\boldsymbol{\omega} \cdot \nabla)\,\mathbf{u}$ in (1.2) in contrast to the global well-posedness of the two-dimensional equations when the vortex stretching term vanishes. In fact, there are a number of literatures about global well-posedness results of the two-dimensional incompressible Euler equations when the initial vorticity $\boldsymbol{\omega}_0$ lies in various function spaces (see [22] and references therein). Specifically, when $\boldsymbol{\omega}_0 \in L^1 \cap L^\infty$, Yudovich [26] proved the global existence and uniqueness of weak solutions. Later, the result was improved by Vishik [24] and Yudovich [27] with $\boldsymbol{\omega}_0$ in a class slightly larger than L^∞. If $\boldsymbol{\omega}_0$ lies in $L^1 \cap L^p$ with $p > 1$, the global existence result was proved by Diperna–Majda [6]. When $\boldsymbol{\omega}_0$ is a finite Radon measure with one sign, the global existence result was studied by Delort [5], Majda [21], Evans–Müller [9] and Liu–Xin [18] via different approaches.

However, when it comes to the three-dimensional case, local well-posedness of classical solutions as studied in [22]. Some global well-posedness results were established when considering periodic domains or the solution with some symmetric property. When the domains are periodic, recently, based on the method introduced in [16,17], Szekelyhidi [23] and Wiedemann [25] constructed the global admissible and L^2 weak solutions respectively. If the flow is axisymmetric and the swirl component of velocity fields vanishes, the global well-posedness results similar to two-dimensional case were extensively studied (see [3,4,13–15,22] and references therein). If the swirl component appears, the global well-posedness issue is still open except [11] for Navier–Stokes equations with a class of large data.

Another attractive issue for global well-posedness issue in three-dimensional flow is helically symmetric case, which means that the flow is invariant under a superposition of a rotation around a fixed axis and a simultaneous translation along the rotation axis directions respectively (see Section 2 for more details). Under this situation, when the helical swirl vanishes (see (2.12) below) and x' lies in bounded domains, the global well-posedness results for smooth/strong solutions corresponding to the initial regular velocity filed/bounded vorticity had been established respectively by Dutrifoy [7] and Ettinger–Titi [8]. For the whole space case, when the third component ω_0 of initial vorticity is compactly supported and belongs to L^p with $p > \frac{4}{3}$, the global weak solutions were obtained by Bronzi–Lopes–Lopes in [2]. In addition, to our best knowledge, there is no any global well-posedness result when the helical swirl exists in three-dimensional case. Along this line, the vanishing viscosity limit for the three-dimensional helically symmetric