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Multiplicity results in the non-coercive case for an elliptic problem with critical growth in the gradient

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Abstract

We consider the boundary value problem

$$-\Delta u = \lambda c(x)u + \mu(x)|\nabla u|^2 + h(x), \qquad u \in H_0^1(\Omega) \cap L^\infty(\Omega), \tag{P_λ}$$

where $\Omega \subset \mathbb{R}^N$, $N \geq 3$ is a bounded domain with smooth boundary. It is assumed that c, h belong to $L^p(\Omega)$ for some $p > N$ with $c \gneqq 0$ as well as $\mu \in L^\infty(\Omega)$ and $\mu \geq \mu_1 > 0$ for some $\mu_1 \in \mathbb{R}$. It is known that when $\lambda \leq 0$, problem (P_λ) has at most one solution. In this paper we study, under various assumptions, the structure of the set of solutions of (P_λ) assuming that $\lambda > 0$. Our study unveils the rich structure of this problem. We show, in particular, that what happen for $\lambda = 0$ influences the set of solutions in all the half-space $]0, +\infty[\times(H_0^1(\Omega) \cap L^\infty(\Omega))$. Most of our results are valid without assuming that h has a sign. If we require h to have a sign, we observe that the set of solutions differs completely for $h \gneqq 0$ and $h \lneqq 0$. We also show when h has a sign that solutions not having this sign may exists. Some uniqueness results of signed solutions are also derived. The paper ends with a list of open problems.

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1. Introduction

We consider the boundary value problem

$$-\Delta u = \lambda c(x)u + \mu(x)|\nabla u|^2 + h(x), \qquad u \in H_0^1(\Omega) \cap L^\infty(\Omega), \tag{P_λ}$$

under the assumption

$$\begin{cases} \Omega \subset \mathbb{R}^N,\ N \geq 3 \ \text{ is a bounded domain with } \partial\Omega \text{ of class } C^{1,1}, \\ c \text{ and } h \text{ belong to } L^p(\Omega) \text{ for some } p > N \text{ and satisfy } c \gneqq 0, \\ \mu \in L^\infty(\Omega) \text{ satisfies } 0 < \mu_1 \leq \mu(x) \leq \mu_2. \end{cases} \tag{A}$$

Depending on the parameter $\lambda \in \mathbb{R}$ we study the existence and multiplicity of solutions of (P_λ). By solutions we mean functions $u \in H_0^1(\Omega) \cap L^\infty(\Omega)$ satisfying

$$\int_\Omega \nabla u \nabla v\, dx = \lambda \int_\Omega c(x)uv\, dx + \int_\Omega \mu(x)|\nabla u|^2 v\, dx + \int_\Omega h(x)v\, dx\,,$$

for any $v \in H_0^1(\Omega) \cap L^\infty(\Omega)$.

First observe that, by the change of variable $v = -u$, problem (P_λ) reduces to

$$-\Delta v = \lambda c(x)v - \mu(x)|\nabla v|^2 - h(x), \qquad v \in H_0^1(\Omega) \cap L^\infty(\Omega).$$

Hence, since we make no assumptions on the sign of h, we actually also consider the case where $|\nabla u|^2$ has a negative coefficient.

The study of quasilinear elliptic equations with a gradient dependence up to the critical growth $|\nabla u|^2$ was essentially initiated by Boccardo, Murat and Puel in the 80's and it has been an active field of research until now. Under the condition $\lambda c(x) \leq -\alpha_0 < 0$ a.e. in Ω for some $\alpha_0 > 0$, which is usually referred to as the *coercive case*, the existence of a unique solution of (P_λ) is guaranteed by assumption (A). This is a special case of the results of [8,9] for the existence and of [6,7] for the uniqueness. Let us point out that the requirement to deal with bounded solutions in (P_λ) is essential to the uniqueness results. Indeed if one admits more general solutions, the existence of infinitely many solutions is known in several cases, see for example [1,2].

The limit case where one just require that $\lambda c(x) \leq 0$ a.e. in Ω is more complex. There had been a lot of contributions [2,13,19,20] when $\lambda = 0$ (or equivalently when $c \equiv 0$) but the general case where $\lambda c \leq 0$ may vanish only on some parts of Ω was left open until the paper [4]. It appears in [4] that under assumption (A) the existence of solutions is not guaranteed, additional conditions are necessary. When $\lambda = 0$ this was already observed in [13]. By [4], the uniqueness itself holds as soon as $\lambda c(x) \leq 0$ a.e. in Ω. See also [5] for a related uniqueness result in a more general frame.

The case $\lambda c \gneqq 0$ remained unexplored until very recently. Following the paper [22] which consider a particular case, Jeanjean and Sirakov [18] study a problem directly connected to (P_λ).