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Almost global existence for the semi-linear Klein–Gordon equation on the circle

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Abstract

We show that the solution to the semi-linear Klein–Gordon equation on the circle with a nonlinearity satisfying a convenient condition, exists almost globally, for almost every positive mass, provided that it is either even or odd as a function of the space variable. We also show, that if the nonlinearity is independent of the space derivative of the unknown and if it is also even as a function of the time derivative, then the solution exists almost globally, for almost every positive mass. The results are based on the method of normal forms. The difficulty is to find a structure of the nonlinearity so that the process of normal forms can be performed up to any order.

MSC: 35L70; 58J45

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1. Introduction

Let (M^d, g) be a d-dimensional Riemannian manifold without boundary. Consider nonlinear Klein–Gordon equations of type

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$$\begin{cases} (\partial_t^2 - \Delta + m^2)v = f(v, \nabla_g v, \partial_t v), & (t,x) \in \mathbb{R}_+^* \times M^d, \\ v|_{t=0} = \epsilon v_0, \quad \partial_t v|_{t=0} = \epsilon v_1, \end{cases} \tag{1.1}$$

where $0 < \epsilon \ll 1$, $v_0, v_1 \in C^\infty(M^d)$, $m > 0$, and where f is a smooth function of $(v, \nabla_g v, \partial_t v)$ vanishing at the origin at least of order 2. Our main results concern long time existence for solutions to such a Cauchy problem on the circle. First of all, let us recall some known results about this problem on various manifolds.

If M^d is the Euclidean space $\mathbb{R}^d$ and if the data is furthermore assumed to be compactly supported, then making use of dispersion, one is able to obtain global solutions when $d \geq 2$ and to show that the solution exists over time intervals of exponential length e^{c/ϵ^2} when $d = 1$. The result is in general optimal, but global existence was proved in $d = 1$ if the nonlinearity satisfies a null condition (a terminology introduced by Klainerman). We refer to the introduction of [11] for references concerning those results. For the results when the data just decay weakly as H^s function, we refer the reader to [9,15] and references therein.

If M^d is a compact manifold, the situation becomes quite different since no dispersion is available for the corresponding linear equation, due to the fact that the Laplace–Beltrami operator on a compact manifold has pure point spectrum. However, in the cases of $\mathbb{S}^d$ ($d \geq 1$) or more generally Zoll manifolds, when the nonlinearity depends only on v and when m stays outside a zero measure subset of $(0, +\infty)$, one may be able to show almost global existence: the solution exists over a time interval of length $c\epsilon^{-N}$ with uniformly bounded H^s norm, for any N and large enough s (see [3,1,2]). Such a result also holds for a similar equation of (1.1) with the nonlinearity f replaced by a quasi-linear one that has a Hamiltonian structure (therefore independent of $\partial_t v$) when M^d is the sphere $\mathbb{S}^d$ (see [4,5]). But for problem (1.1) with a nonlinearity f depending on all derivatives of first order, in particular $\partial_t v$, only a weaker result is known in the cases of $\mathbb{S}^1$ and more generally Zoll manifolds (see [6,10]). We ask the question, as in the case of the Euclidean space, of finding a "null condition" for nonlinearities $f(v, \nabla_g v, \partial_t v)$, which would allow almost global existence of small H^s solutions for almost every $m > 0$. This is actually one of the unsolved questions proposed in [2]. In this paper, we give a partial answer to this question in the case of $\mathbb{S}^1$. We shall use the method of normal forms, which was initially introduced in the framework of nonlinear Klein–Gordon equations in [14]. The novelty is to find a structure of the nonlinearity so that the normal form process can be carried out up to any order.

Let us explain the main idea briefly. We want to control the Sobolev energy of the solutions

$$\frac{d}{dt}\left(\|v(t,\cdot)\|_{H^{s+1}}^2 + \|\partial_t v(t,\cdot)\|_{H^s}^2\right). \tag{1.2}$$

Using the equation, this quantity may be written as a sum of multi-linear expressions in v, $\partial_t v$, having a special property coming from the structure of the nonlinearity. For the multi-linear expression homogeneous of k in (1.2), we perturb the Sobolev energy by an expression of homogeneous of degree k such that

- its time derivatives cancel out the main contribution, up to remainders of higher order;
- the perturbation is bounded by powers of $\|v(t,\cdot)\|_{H^{s+1}} + \|\partial_t v(t,\cdot)\|_{H^s}$.

The difficulty is to find the special structure of the nonlinearity so that

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