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Construction of response solutions for two classes of quasi-periodically forced four-dimensional nonlinear systems with degenerate equilibrium point under small perturbations ☆

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Abstract

By developing two KAM theorems, in this paper, we show that two classes of quasi-periodically forced four-dimensional nonlinear systems with degenerate equilibrium point, a reversible system and a non-conservative system, admit a response solution under small perturbations. For the degenerate reversible system, applying special structure of unperturbed nonlinear term and Herman method, we successfully control the shift of equilibrium point, which is difficult in view of the degenerate linear term. For the degenerate non-conservative system, KAM method is brought into force even in completely degenerate case because of the restrictions on the smallness and average of perturbation. Moreover, arithmetic condition on the frequency is assumed to satisfy the Brjuno–Rüssmann's non-resonant condition. By the Pöschel–Rüssmann KAM method, we prove that these two kinds of perturbed systems can be reduced to a suitable normal form with zero as equilibrium point by a quasi-periodic transformation.

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1. Introduction

In 1950, J.J. Stoker ([1]) considered a nonlinear oscillator with damping and quasi-periodic forcing, of the form

$$\ddot{x} + c\dot{x} + x - \beta x^2 = \epsilon f(\omega t), \tag{1.1}$$

in which f is a quasi-periodic function with frequency vector $\omega = (\omega_1, \dots, \omega_d) \in \mathbb{R}^d$. By using a contraction argument, he showed that if either for given ϵ and damping strength c is sufficiently large or given c and the ϵ is sufficiently small, the above system has a quasi-periodic solution $x(t) = \varphi(\omega t)$ with the same frequency as the forcing f, so-called *response solution*. This question, of whether response solutions exist in equations like (1.1) for c close to 0, is nowadays called 'Stoker's problem'. There are two the pioneering works in this subject: J. Moser ([2]) solved Stoker's problem for reversible systems, B.L.J. Braaksma and H.W. Broer ([3]) considered Stoker's problem for families of general nonlinear oscillators.

This problem has received wide attention starting from the above two works. For example, see [4–8] for Hamiltonian systems and see [9–12] for dissipative systems. It is worth to point out that the existence of response solutions for a general parametrized Hamiltonian has been proved, for real analytic nonlinearities, by H.W. Broer, H. Hanßmann, A. Jorba, J. Villanueva, and F.O.O. Wagener ([8]) via normal form techniques.

For quasi-periodic systems, all works mentioned above mainly concerned with the perturbations of system with non-degenerate equilibrium points. In particular, A. Jorba and C. Simó ([12]) considered the following nonlinear quasi-periodic system

$$\dot{x} = Ax + h(x,t) + f(x,t,\varepsilon), \quad x \in \mathbb{R}^n, \tag{1.2}$$

where A is assumed to be elliptic, that is, all eigenvalues are purely imaginary and nonzero, $h = \mathcal{O}(x^2)$ as $x \to 0$ and f is a small perturbation. They proved that under some non-resonance conditions and some non-degeneracy conditions, there exists a non-empty Cantor subset $\mathcal{E} \subset [0, \varepsilon_0]$, $\varepsilon_0 > 0$, such that the system (1.2) can be reduced by an affine quasi-periodic transformation to the following form:

$$\dot{x} = A^*(\varepsilon)x + h^*(x,t,\varepsilon), \quad x \in \mathbb{R}^n,$$

where $A^*(\varepsilon)$ is a constant matrix close to A and $h^*(x,t,\varepsilon) = \mathcal{O}(x^2)$ $(x \to 0)$ is a high-order term. Therefore, the system (1.2) has a response solution near the zero equilibrium point. In [12], the linear part of systems requires to be non-degenerate. If the equilibrium point is degenerate, that is, A has zero as an eigenvalue, this problem becomes more difficulty because the linear term cannot control the shift of equilibrium point very well. The result in [12] cannot be applied directly to the degenerate case. This shows that new technique and more restrictions on high-order term and perturbation term are necessary to study the degenerate case. With the development of the KAM theory, there are already some important results on degenerate lower-dimensional invariant tori for Hamiltonian systems [13–15]. In the last years, great attention has been paid to study