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Homogeneous solutions of extremal Pucci's equations in planar cones

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Abstract

We derive explicit expressions of the homogeneous solutions with constant sign in two dimensional cones for Pucci's extremal equations. As examples of possible applications, we obtain monotonicity formulas for all nonnegative supersolutions and necessary and sufficient explicit conditions for non-existence results of Liouville type.

Keywords: Fully nonlinear elliptic equations; Pucci's extremal operators; Homogeneous solutions; Cones

1. Introduction

The goal of the present paper is to determine explicit solutions in two dimensional cones of homogeneous Dirichlet boundary value problems associated with the extremal elliptic equations

$$\mathcal{M}^{\pm}_{\lambda,\Lambda}(D^2u) = 0\,, \tag{1.1}$$

where $\mathcal{M}^{+}_{\lambda,\Lambda}$ and $\mathcal{M}^{-}_{\lambda,\Lambda}$ are the Pucci's extremal operators introduced by C. Pucci in [14].

We recall that, given two ellipticity constants $\Lambda \geq \lambda > 0$, the operator $\mathcal{M}^{+}_{\lambda,\Lambda}$, as a function of $M \in \mathcal{S}_n$ where $\mathcal{S}_n$ is the set of $n \times n$ symmetric matrices, can be equivalently defined either as

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$$\mathcal{M}^{+}_{\lambda,\Lambda}(M) = \Lambda \sum_{\mu_i>0} \mu_i + \lambda \sum_{\mu_i<0} \mu_i$$

where $\mu_1, \ldots, \mu_n$ are the eigenvalues of M, or as

$$\mathcal{M}^{+}_{\lambda,\Lambda}(M) = \sup_{A\in\mathcal{A}_{\lambda,\Lambda}} \operatorname{tr}(AM)$$

with $\mathcal{A}_{\lambda,\Lambda} = \{A \in \mathcal{S}_n : \lambda I_n \leq A \leq \Lambda I_n\}$, where I_n is the identity matrix in $\mathcal{S}_n$ and $\leq$ is the usual partial ordering in $\mathcal{S}_n$, meaning that $A \leq B$ if and only if $A - B$ has nonpositive eigenvalues.

The operator $\mathcal{M}^{-}_{\lambda,\Lambda}$ is likewise defined either as

$$\mathcal{M}^{-}_{\lambda,\Lambda}(M) = \lambda \sum_{\mu_i>0} \mu_i + \Lambda \sum_{\mu_i<0} \mu_i$$

or as

$$\mathcal{M}^{-}_{\lambda,\Lambda}(M) = \inf_{A\in\mathcal{A}_{\lambda,\Lambda}} \operatorname{tr}(AM) ,$$

and it is easy to verify that

$$\mathcal{M}^{+}_{\lambda,\Lambda}(M) = -\mathcal{M}^{-}_{\lambda,\Lambda}(-M) \quad \text{for all } M \in \mathcal{S}_n . \tag{1.2}$$

The Pucci operators $\mathcal{M}^{\pm}_{\lambda,\Lambda}$ are extremal not only in the class of linear elliptic operators, but they satisfy

$$\mathcal{M}^{-}_{\lambda,\Lambda}(M) \leq F(x, M) \leq \mathcal{M}^{+}_{\lambda,\Lambda}(M) \quad \text{for all } M \in \mathcal{S}_n$$

for any uniformly elliptic operator F having ellipticity constants Λ and λ and satisfying $F(x, O) = 0$. This explains the central role played by $\mathcal{M}^{\pm}_{\lambda,\Lambda}$ in the elliptic theory, see the monograph [6], and the importance of finding explicit solutions of (1.1), which act as barrier functions in comparison theorems for solutions of general fully nonlinear equations.

In particular, solutions of (1.1) in cone-like domains may be used in the asymptotic analysis near corners of solutions in Lipschitz domains of general elliptic equations with asymptotically negligible lower order terms, as performed for divergence form operators, see e.g. [7].

Equations (1.1) are known to have homogeneous solutions in cone-like domains of $\mathbb{R}^n$ for any dimension $n \geq 2$, see [2,10,12]. In particular, the more recent results of [2] establish that, for any positively homogeneous and uniformly elliptic operator F and any cone $\mathcal{C} \subset \mathbb{R}^n$, the homogeneous Dirichlet problem

$$\begin{cases} F(D^2u) = 0 & \text{in } \mathcal{C} \\ u = 0 & \text{on } \partial\mathcal{C} \setminus \{0\} \end{cases} \tag{1.3}$$

has exactly four (up to normalization) constant sign homogeneous solutions, that are of the form