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Blow-up analysis and existence results in the supercritical case for an asymmetric mean field equation with variable intensities

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Abstract

A class of equations with exponential nonlinearities on a compact Riemannian surface is considered. More precisely, we study an asymmetric sinh-Gordon problem arising as a mean field equation of the equilibrium turbulence of vortices with variable intensities.

We start by performing a blow-up analysis in order to derive some information on the local blow-up masses. As a consequence we get a compactness property in a supercritical range.

We next introduce a variational argument based on improved Moser–Trudinger inequalities which yields existence of solutions for any choice of the underlying surface.

MSC: 35J61; 35J20; 35R01; 35B44

Keywords: Geometric PDEs; Mean field equation; Blow-up analysis; Variational methods

1. Introduction

We consider here the following equation

$$-\Delta u = \rho_1 \left(\frac{h_1 e^u}{\int_M h_1 e^u \, dV_g} - \frac{1}{|M|} \right) - a\rho_2 \left(\frac{h_2 e^{-au}}{\int_M h_2 e^{-au} \, dV_g} - \frac{1}{|M|} \right), \tag{1}$$

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where $a \in (0,1)$, h_1, h_2 are smooth positive functions, ρ_1, ρ_2 are two positive parameters and (M, g) is a compact orientable surface with no boundary equipped with a Riemannian metric g. For the sake of simplicity, we normalize the total volume of M so that $|M| = 1$.

Equation (1) arises in the context of the statistical mechanics description of $2D$-turbulence: the physical model was first introduced in [31] and different mean field equations have been obtained according to different constraints. In the case that the circulation number density is subject to a probability measure, under a *deterministic* assumption on the vortex intensities, the model is ruled by the following equation, see [36]:

$$-\Delta u = \rho \int\limits_{[-1,1]} \alpha \left(\frac{e^{\alpha u}}{\int_M e^{\alpha u}\, dV_g} - \frac{1}{|M|} \right) \mathcal{P}(d\alpha), \tag{2}$$

where u denotes the stream function of a turbulent Euler flow, $\mathcal{P}$ is a Borel probability measure defined on the interval $[-1, 1]$ describing the point vortex intensity distribution and $\rho > 0$ is a physical constant related to the inverse temperature. Equation (1) is related to the latter model for the particular choice $\mathcal{P}(d\alpha) = \tau_1 \delta_1(d\alpha) + \tau_a \delta_{-a}(d\alpha)$, where $a \in (0, 1)$ and τ_1, τ_a are positive parameters such that $\tau_1 + \tau_a = 1$. Observe that we focus just one the different-sign problem since the case $supp\, \mathcal{P} \subset [0, 1]$ presents some differences and it is considered in [18].

In order to describe the nature of equation (1) and the strategy to attack it, let us first consider the standard mean field equation obtained from (2) with $\mathcal{P}(d\alpha) = \delta_1$, namely

$$-\Delta u = \rho \left(\frac{h\, e^{u}}{\int_M h\, e^{u}\, dV_g} - \frac{1}{|M|} \right). \tag{3}$$

The latter equation has been widely studied since it is related to the prescribed Gaussian curvature problem [1,5,6,20,37] and to the mean field equation of Euler flows [4,19]. For a survey of the latter equation we refer to [26,38].

One of the main difficulties in dealing with this class of equations is due to the loss of compactness, as its solutions might blow-up. As a consequence, the first step is to analyze the bubbling phenomenon. We point out an important property that was observed for (3), see [3,21,22]: for a sequence of blow-up solutions $\{u_k\}_k$ to (3) relative to ρ_k with blow-up point $\bar{x}$ the following quantization holds true

$$\tilde{\sigma}(\bar{x}) = \lim_{\delta \to 0} \lim_{k \to +\infty} \rho_k \frac{\int_{B_\delta(\bar{x})} h\, e^{u_k}}{\int_M h\, e^{u_k}\, dV_g} = 8\pi. \tag{4}$$

The latter property yields important consequences in many applications, in particular for what concerns compactness results, see the discussion later on.

In the more general situation of (1) (and (2)) the blow-up analysis has still to be completed. We refer to [15,29,30,28,34] for the progress in this direction. We stress that for $a = 1$ equation (1) reduces to the sinh-Gordon problem

$$-\Delta u = \rho_1 \left(\frac{h_1 e^{u}}{\int_M h_1 e^{u}\, dV_g} - \frac{1}{|M|} \right) - \rho_2 \left(\frac{h_2 e^{-u}}{\int_M h_2 e^{-u}\, dV_g} - \frac{1}{|M|} \right), \tag{5}$$

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