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Nonintegrability of dynamical systems with homo- and heteroclinic orbits ☆

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Abstract

We consider general n-dimensional systems of differential equations having an $(n-2)$-dimensional, locally invariant manifold on which there exist equilibria connected by heteroclinic orbits for $n \geq 3$. The system may be non-Hamiltonian and have no saddle-centers, and the equilibria are allowed to be the same and connected by a homoclinic orbit. Under additional assumptions, we prove that the monodromy group for the *normal variational equation*, which is represented by components of the *variational equation* normal to the locally invariant manifold and defined on a Riemann surface, is diagonalizable or infinitely cyclic if the system is real-meromorphically integrable in the meaning of Bogoyavlenski. We apply our theory to a three-dimensional volume-preserving system describing the streamline of a steady incompressible flow with two parameters, and show that it is real-meromorphically nonintegrable for almost all values of the two parameters.

1. Introduction

Nonintegrability of differential equations is one of the most important topics in dynamical systems [9,10,20]. It is believed that complicated dynamical behavior such as chaos may occur in general if differential equations are nonintegrable [10], although there are several notions of

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integrability. However, it is difficult in general to determine whether given differential equations are integrable or not.

For Hamiltonian systems, the notion of integrability has been established and called the Liuoville integrability [18]: an n-degree-of-freedom Hamiltonian system is said to be *Liuoville integrable* if it possesses n independent first integrals 'in involution'. It is a well-known result as the Liouville–Arnold theorem that if an n-degree-of-freedom Hamiltonian system is integrable in this sense and a level set of n first integrals is compact, then the flow on the level set is diffeomorphic to a linear flow on the n-dimensional torus $\mathbb{T}^n$, i.e., it is quasiperiodic [2]. Much research has been done on nonintegrability of Hamiltonian systems [9,10,20,21], and two powerful techniques to prove their nonintegrability have been developed: the Ziglin analysis [33] and Morales–Ramis theory [20,23]. Both the techniques rely on some properties of *variational equations* (VEs), i.e., linearized equations, along particular solutions of the Hamiltonian systems. The former uses their monodromy matrices and the latter uses their differential Galois groups [8,13,29].

For general differential equations which may be non-Hamiltonian, Bogoyavlenski [6] introduced a general striking definition of integrability, among several notions (see, e.g., [21]). Consider differential equations of the form

$$\dot{x} = f(x), \quad x \in \mathbb{R}^n, \tag{1}$$

where $f : \mathbb{R}^n \to \mathbb{R}^n$ is analytic. His definition of integrability is stated for (1) as follows.

Definition 1.1 *(Bogoyavlenski).* Equation (1) is called $(q, n-q)$-*integrable* or just *integrable* if there exist q (≥ 1) vector fields $f_1(x)(:= f(x))$, $f_2(x), \dots, f_q(x)$ and $n-q$ scalar-valued functions $H_1(x), \dots, H_{n-q}(x)$ such that the following three conditions hold:

(i) $f_1, \dots, f_q$ are linearly independent and $DH_1, \dots, DH_{n-q}$ are linearly independent almost everywhere;
(ii) $f_1, f_2, \dots, f_q$ commute, i.e., $[f_j, f_k] := (Df_k)f_j - (Df_j)f_k = 0$, for $j, k = 1, \dots, q$;
(iii) $H_1, \dots, H_{n-q}$ are first integrals of $f_1, f_2, \dots, f_q$, i.e., $(DH_k)f_j = 0$ for $j = 1, \dots, q$ and $k = 1, \dots, n-q$.

If $f_1, f_2, \dots, f_q$ and $H_1, \dots, H_{n-q}$ are real-meromorphic, then equation (1) is said to be *real-meromorphically integrable*.

Definition 1.1 is thought to be the most general at present since equation (1) can be integrable even when it has only $n-q$ $(< n-1)$ first integrals for some $q > 1$. In particular, if a Hamiltonian system is Liouville integrable, then it is also integrable in the meaning of Bogoyavlenski. Ayoul and Zung [3] extended the Morales–Ramis theory to the nonintegrability of non-Hamiltonian differential equations in this meaning. Their method was successfully applied to prove the nonintegrability of the five-dimensional Karabut system, which is non-Hamiltonian and appears in relation to a fluid of finite depth (e.g., [14]), in [7].

On the other hand, Morales-Ruiz and Peris [22] studied a class of two-degree-of-freedom Hamiltonian systems with saddle-center equilibria connected by homoclinic orbits, and applied the Morales–Ramis theory to obtain a sufficient condition for their nonintegrability. Moreover, they used the results of [11,17] to show that chaotic dynamics occurs if the condition holds. Their result was extended to more general two-degree-of-freedom Hamiltonian systems with saddle-centers connected by homoclinic orbits in [32], based on the result of [31] as well as [22].

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