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Asymptotic stability estimates near an equilibrium point

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Abstract

We use the error bounds for adiabatic invariants found in the work of Chartier, Murua and Sanz-Serna [3] to bound the solutions of a Hamiltonian system near an equilibrium over exponentially long times. Our estimates depend only on the linearized system and not on the higher order terms as in KAM theory, nor do we require any steepness or convexity conditions as in Nekhoroshev theory. We require that the equilibrium point where our estimate applies satisfy a type of formal stability called Lie stability.

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Keywords: Hamiltonian systems; Stability; Adiabatic invariants; Asymptotic estimates

1. Introduction

In our previous work [15] we started a search for stability results around an equilibrium point that depend only on the quadratic part of the Hamiltonian, i.e., only on the linearized system. We do not search for stability criteria that depend on the higher order terms as in KAM theory, or on the steepness and convexity conditions found in Nekhoroshev theory. The only complete result of

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this type is Dirichlet's Theorem [4], which gives stability of the equilibrium when the quadratic part is positive (or negative) definite. Stability cannot be determined from the eigenvalues of the linearized system alone as shown by the classical example of Cherry – see [14]. This gave rise to many different formal results, as discussed in [15,21,22] and the references therein.

Here we take a different approach. We establish bounds for exponentially long times on the actual solutions near an equilibrium of an analytic Hamiltonian system following the tradition found in [7,8] – see Lochak [12,13]. First we use the theory developed by dos Santos and coworkers [21,22] on Lie stable systems to prepare the quadratic part of the Hamiltonian to obtain enough proper formal adiabatic invariants. Next the error bounds found in Chartier, Murua and Sanz-Serna [3] are applied to the adiabatic invariants. And finally, a straightforward Liapunov type argument transfers the estimates from the adiabatic invariants to the actual solutions.

Other approaches to get estimates for elliptic equilibria of Hamiltonian systems may be found in [19,6,17,20]. Similarly to Lochak [12,13], the authors of the previous papers apply Nekhoroshev theory and obtain sharp results on stability over exponentially long times. In their papers they require convexity of the Hamiltonian (or related conditions), hence there are no resonances of order less than 5, but they do not assume any Diophantine conditions among the frequencies. Moreover they do not assume any type of formal stability. Our approach is different, since we provide estimates for elliptic equilibria that are formally stable and the kind of stability required is characterized by the quadratic terms of the Hamiltonian. We also require a Diophantine hypothesis among some of the frequencies of the linearized equations of motion.

The present paper has six sections. In Section 2 we state our main theorem and give an example. In Section 3 we deal with the calculation of formal invariants for a resonant Hamiltonian using a normal form approach. In addition we present two propositions which characterize Hamiltonians that are Lie stable by checking only the linearized equation, relating the concept of Lie stability to the existence of a linear combination of the formal integrals for the normal form Hamiltonian. Since our theory relies on the estimates for adiabatic invariants of Chartier et al., in Section 4 we present these authors' result and its connection with our approach. In Section 5 we give the proof of our main theorem. Finally, in Section 6 we relate Lie stability to normal stability and apply the ideas of the previous sections to the spatial case of the circular restricted three body problem.

2. The system

We consider a real analytic Hamiltonian defined in $\mathcal{N}$, a neighborhood of the origin in $\mathbb{R}^{2n}$, of the form

$$\mathcal{H}(x) = \mathbb{H}(x) + \mathcal{K}(x)\,, \tag{1}$$

whose equations of motion are the Hamiltonian system

$$\dot{x} = \mathcal{J}\nabla\mathcal{H}(x)\,, \tag{2}$$

where $\mathcal{J}$ is the standard $2n \times 2n$ symplectic matrix of Hamiltonian theory [14].

In (1) above, $\mathbb{H}$ is the quadratic Hamiltonian

$$\mathbb{H}(x) = \tfrac{1}{2}x^T Sx\,, \tag{3}$$