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Asymptotically dichotomic almost periodic differential equations

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Abstract

Consider a non-linear differential equation in $\mathbb{R}^N$ which asymptotically behaves as a linear equation admitting an exponential dichotomy. We wonder if almost periodic solutions exist when we add to the equation an almost periodic forcing term, large enough and not vanishing too much. A positive answer has been given in [3] for the scalar case $N = 1$ and our aim is to extend that result to higher dimensions. We discover that the extension seems to be driven by a new ingredient, namely the type of the exponential dichotomy: besides the pure stable types, the mixed hyperbolic type is now possible and leads to a weaker than expected extension. An example shows that a stronger extension cannot be obtained by the same method. The approach is blended and mixes methods of differential equations and functional analysis, especially when estimating norm and spectral radius of some crucial positive but non-compact linear integral operators.

Keywords: Almost periodic solutions; Exponential dichotomies; Contractions and Fixed points; Positive operators; Spectral radius; Haar measure

1. Introduction

Consider the non-linear equation in $\mathbb{R}^N$:

$$\dot{x} + A(t)x = f(x) + \lambda h(t) \tag{1.1}$$

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where the coefficients matrix A and the forcing term h depend almost periodically on time and λ is a scalar parameter. We suppose that the linear equation:

$$\dot{x} + A(t)x = 0 \tag{1.2}$$

has an exponential dichotomy and that the non-linearity f is a bounded and Lipschitz continuous function on $\mathbb{R}^N$.

The case $f = 0$ is trivial, inasmuch equation (1.1) has a unique bounded solution, which is automatically almost periodic, for every λ: see [4] for a proof. A similar conclusion holds also when the non-linearity is small in a suitable sense. More precisely, standard roughness results for exponential dichotomies (see chapter 8 of [5] or [4]) apply to cover the case where the essential norm of the derivative:

$$\|f'\|_\infty = \operatorname*{ess\text{-}sup}_{x\in\mathbb{R}^N} |f'(x)|$$

is small enough. On the contrary, no global restrictions on the derivative will be considered here. Our assumptions on the non-linearity are discussed in Section 2 and amount to require that f is bounded while f' is small at infinity only:

$$\lim_{x\to\infty} f'(x) = 0\,.$$

The aim is to find, when the parameter λ is large enough, existence results for almost periodic solutions to equation (1.1). The same problem has been already considered in [3] but for an equation (1.2) which is *scalar* and *autonomous*. The investigation there was intended to overcome the global monotonicity assumptions which are used in a large part of the literature: see for instance [5,1] and [2] and the references therein. However, overcoming the monotonicity seems not to be always possible: see for instance [14,6] and [15]. Some price to pay on the forcing term is expected: roughly speaking, the result proved in [3] says that everything works fine for λ large when the forcing term h “does not vanish too much”.

To give a more precise description, we need to introduce a couple of ingredients. Imagine first that a metrizable, compact and abelian topological group Ω is given, together with a continuous homomorphism $\Psi : \mathbb{R} \to \Omega$ with dense image. Imagine moreover that both the almost periodic functions A and h are *representable* over the pair (Ω, Ψ), in the sense that there exist two continuous functions $\mathfrak{A}$ and $\mathfrak{h}$ on Ω satisfying:

$$A(t) = \mathfrak{A}(\Psi(t)) \qquad h(t) = \mathfrak{h}(\Psi(t)) \tag{1.3}$$

for every t. This is a standard assumption in the literature, related to the notion of *hull* of an almost periodic function: see Appendix B for more details. For instance, the choice $\Omega \cong \mathbb{S}^1$ is suitable to represent periodic functions only, while aperiodic almost periodic functions need more complicate groups like torii or solenoids. Coming back to a general Ω, as any other compact topological group it supports a unique Haar measure: let us denote it $\mathfrak{m}$. This measure is the key ingredient in [3] inasmuch the authors show that:

$$\mathfrak{m}\left(\mathfrak{h}^{-1}(0)\right) = 0 \tag{1.4}$$