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Positive solutions for nonlinear elliptic problems with dependence on the gradient

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Abstract

We consider a quasilinear Neumann problem with a differential operator and a reaction term, both dependent on u and Du. Using topological methods together with suitable truncation and comparison techniques, we show that the problem has at least one positive smooth solution.

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1. Introduction

Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain with a C^2-boundary $\partial\Omega$. In this paper, we deal with the following quasilinear elliptic Neumann problem:

$$\begin{cases} -\operatorname{div}(a(u(z))Du(z)) + \beta(z)u(z) = f(z, u(z), Du(z)) & \text{in } \Omega, \\ \frac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega, \ u \geqslant 0. \end{cases} \tag{1.1}$$

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Elliptic problem with the reaction term depending on the gradient was studied using a variety of methods. Indicatively we mention the pioneering works of Amann–Crandall [1], Brézis–Turner [2], Ruiz [15], Yan [16] who use topological methods and those of de Figueiredo–Girardi–Matzeu [4], Girardi–Matzeu [8] who use variational methods on the equation with frozen the gradient entry in the reaction term and then employing an iterative technique to pass to the limit to generate a solution of the original equation. In our case, the form of the differential operator which depends on both u and Du, precludes the use of variational methods. Instead we employ topological methods combined with suitable truncation and comparison techniques. Our work here appears to be the first one dealing with Neumann problems. All the aforementioned papers examine Dirichlet problems.

In the next section we prepare the ground for the analysis of the problem (1.1) by recalling some notions and results which we will need in the sequel and also by proving some auxiliary propositions.

2. Background material

Let X be a reflexive Banach space. By X^* we denote its topological dual and by $\langle \cdot, \cdot \rangle$ the duality brackets for the pair (X^*, X).

Definition 2.1. A map $A\colon X \longrightarrow X^*$ is said to be *pseudomonotone*, if for every sequence $\{u_n\}_{n\geqslant 1} \subseteq X$ such that

$$u_n \xrightarrow{w} u \quad \text{in } X, \qquad A(u_n) \xrightarrow{w} u^* \quad \text{in } X^*$$

and

$$\limsup_{n\to+\infty} \langle A(u_n), u_n - u \rangle \leqslant 0,$$

we have

$$u^* = A(u) \quad \text{and} \quad \langle A(u_n), u_n \rangle \longrightarrow \langle A(u), u \rangle.$$

Also we say that A is *strongly coercive*, if

$$\frac{\langle A(u), u \rangle}{\|u\|} \longrightarrow +\infty \quad \text{as } \|u\| \to +\infty.$$

Remark 2.2. We know that if $A\colon X \longrightarrow X^*$ is monotone, hemicontinuous, then A is pseudomonotone.

Theorem 2.3. *If $A\colon X \longrightarrow X^*$ is pseudomonotone and strongly coercive, then A is surjective.*

The *Leray–Schauder alternative principle* is an effective tool to find fixed points of nonlinear maps.