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Existence theorems for a general 2×2 non-Abelian Chern–Simons–Higgs system over a torus

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Abstract

In this paper we study a general 2×2 non-Abelian Chern–Simons–Higgs system of the form

$$\Delta u_i + \frac{1}{\varepsilon^2}\left(\sum_{j=1}^{2} K_{ji}\mathrm{e}^{u_j} - \sum_{j=1}^{2}\sum_{k=1}^{2} K_{kj}K_{ji}\mathrm{e}^{u_j}\mathrm{e}^{u_k}\right) = 4\pi \sum_{j=1}^{N_i} \delta_{p_{ij}}(x), \quad i = 1, 2$$

over a flat 2-torus $\mathbb{T}^2$, where $\varepsilon > 0$, δ_p is the Dirac measure at p, $N_i \in \mathbb{N}$ $(i = 1, 2)$, K is a non-degenerate 2×2 matrix of the form $K = \begin{pmatrix} 1+a & -a \\ -b & 1+b \end{pmatrix}$, which may cover the physically interesting case when K is a Cartan matrix (of a rank 2 semisimple Lie algebra). Concerning the existence results of this type system over $\mathbb{T}^2$, usually in the literature there is a requirement that $a, b > 0$. However, it is an open problem so far for the solvability about such system with $a, b < 0$, which naturally appears in several Chern–Simons–Higgs models with some specific gauge groups. We partially solve this problem by showing that there exists a constant $\varepsilon_0 > 0$ such that this system admits a solution over the torus if $0 < \varepsilon < \varepsilon_0$ provided $|a|, |b|$ are suitably small. Furthermore, if $ab \geq 0$ in addition, with suitable condition on a, b, N_1, N_2, this system

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admits a mountain-pass solution. Our argument is based on a perturbation approach and the mountain-pass lemma.

Keywords: Chern–Simons vortices; Self-dual equations; Topological solutions; Doubly periodic solutions

1. Introduction

Since 1980's the Chern–Simons terms [12,13] have been used in $2+1$ dimensional gauge field models for the characterization of dually charged vortices [16–19,46], which have applications in many branches of modern physics such as high-temperature superconductivity [39,44], integer and fractional quantum Hall effects [23,48,49], and anyon physics [25,56,57]. For the full Chern–Simons–Higgs models usually it is hard to study the equations of motion due to their complicated structures, even for the radially symmetric case, which was just solved not long ago in [9]. Thanks to the seminal works [32,36], self-dual equations [5,47] have been found in various Abelian and non-Abelian Chern–Simons–Higgs models, relativistic or non-relativistic [20–22,35–37,40–42], which lead to considerable progress for understanding of these equations both physically and mathematically. See [33] for a recent review about Chern–Simons models.

For the Abelian Chern–Simons–Higgs model, the self-dual equations found in [32,36] can be formulated into

$$\Delta u + \frac{1}{\varepsilon^2}\mathrm{e}^u(1-\mathrm{e}^u) = 4\pi\sum_{s=1}^{N}\delta_{p_s}, \tag{1.1}$$

where $\varepsilon > 0$ is a coupling parameter, δ_p denotes the Dirac measure concentrated at point p, and $N \in \mathbb{N}$. Finite-energy condition implies two kinds of admissible boundary conditions on $\mathbb{R}^2$: $u \to 0$ and $u \to -\infty$ at infinity, which are separately called topological and non-topological [59]. The existence of topological solutions for (1.1) was established in [55] by a variational argument, and in [50] via a monotone iteration approach, respectively. For non-topological solutions of (1.1), the first result is due to [51], dealing with the radially symmetric solutions by a shooting argument, which was refined in [11] to tackle more general problems. Concerning the existence of non-radially non-topological solutions, [7] established the first existence result by a perturbation argument, which was extended by [8] and [15] to get more general existence results. Another type physically interesting solutions for (1.1) is called vortex condensates [1] modelling the lattice structure, that is, to construct solutions for (1.1) over a doubly periodic domain (a flat torus), which were first constructed by [6] and later generalized by [52] to get a multiple existence result. More complete existence results concerning (1.1) can be found in the monographs [54,59].

As for the non-Abelian Chern–Simons–Higgs model, the self-dual equations in [20–22] can be reduced into the following nonlinear elliptic system [58,59]

$$\Delta u_i + \frac{1}{\varepsilon^2}\left(\sum_{j=1}^{r}K_{ji}\mathrm{e}^{u_j} - \sum_{j=1}^{r}\sum_{k=1}^{r}K_{kj}K_{ji}\mathrm{e}^{u_j}\mathrm{e}^{u_k}\right) = 4\pi\sum_{j=1}^{N_i}\delta_{p_{ij}}, \quad i=1,\cdots,r, \tag{1.2}$$