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Scattering for the radial 3D cubic focusing inhomogeneous nonlinear Schrödinger equation

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Abstract

The purpose of this work is to study the 3D focusing inhomogeneous nonlinear Schrödinger equation

$$iu_t + \Delta u + |x|^{-b}|u|^2 u = 0,$$

where $0 < b < 1/2$. Let Q be the ground state solution of $-Q + \Delta Q + |x|^{-b}|Q|^2 Q = 0$ and $s_c = (1 + b)/2$. We show that if the radial initial data u_0 belongs to $H^1(\mathbb{R}^3)$ and satisfies $E(u_0)^{s_c} M(u_0)^{1-s_c} < E(Q)^{s_c} M(Q)^{1-s_c}$ and $\|\nabla u_0\|_{L^2}^{s_c} \|u_0\|_{L^2}^{1-s_c} < \|\nabla Q\|_{L^2}^{s_c} \|Q\|_{L^2}^{1-s_c}$, then the corresponding solution is global and scatters in $H^1(\mathbb{R}^3)$. Our proof is based in the ideas introduced by Kenig–Merle [1] in their study of the energy-critical NLS and Holmer–Roudenko [2] for the radial 3D cubic NLS.

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Keywords: Inhomogeneous nonlinear Schrödinger equation; Cauchy problem; Global well-posedness; Scattering

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1. Introduction

In this paper, we consider the Cauchy problem, also called the initial value problem (IVP), for the focusing inhomogeneous nonlinear Schrödinger (INLS) equation on $\mathbb{R}^3$, that is

$$\begin{cases} i\partial_t u + \Delta u + |x|^{-b}|u|^2 u = 0, & t \in \mathbb{R},\ x \in \mathbb{R}^3, \\ u(0,x) = u_0(x), \end{cases} \tag{1.1}$$

where $u = u(t,x)$ is a complex-valued function in space-time $\mathbb{R} \times \mathbb{R}^3$ and $0 < b < 1/2$.

Before reviewing some results about the Cauchy problem (1.1), let us recall the critical Sobolev index. For a fixed $\delta > 0$, the rescaled function $u_\delta(t,x) = \delta^{\frac{2-b}{2}} u(\delta^2 t, \delta x)$ is solution of (1.1) if and only if $u(t,x)$ is a solution. This scaling property gives rise to a scale-invariant norm. Indeed, computing the homogeneous Sobolev norm of $u_\delta(0,x)$, we get

$$\|u_\delta(0,.)\|_{\dot{H}^s} = \delta^{s-\frac{3}{2}+\frac{2-b}{2}} \|u_0\|_{\dot{H}^s}.$$

Thus, the scale invariant Sobolev space is $H^{s_c}(\mathbb{R}^3)$, where $s_c = \frac{1+b}{2}$ (the critical Sobolev index). Note that, the restriction $0 < b < 1/2$ implies $0 < s_c < 1$ and therefore we are in the mass-supercritical and energy-subcritical case. In addition, we recall that the INLS equation has the following conserved quantities

$$M[u_0] = M[u(t)] = \int_{\mathbb{R}^3} |u(t,x)|^2 dx \tag{1.2}$$

and

$$E[u_0] = E[u(t)] = \frac{1}{2}\int_{\mathbb{R}^3} |\nabla u(t,x)|^2 dx - \frac{1}{4}\left\| |x|^{-b}|u|^4 \right\|_{L^1_x}, \tag{1.3}$$

which are Mass and Energy, respectively.

Next, we briefly review recent developments on the well-posedness theory for the general INLS equation

$$\begin{cases} i\partial_t u + \Delta u + |x|^{-b}|u|^\alpha u = 0, & x \in \mathbb{R}^N, \\ u(0,x) = u_0(x). \end{cases} \tag{1.4}$$

Genoud and Stuart [3,4], using the abstract theory developed by Cazenave [5] and some sharp Gagliardo–Nirenberg inequalities, showed that (1.4) is well-posed in $H^1(\mathbb{R}^N)$

- locally if $0 < \alpha < 2^*$,
- globally for small initial condition if $\frac{4-2b}{N} < \alpha < \frac{4-2b}{N-2}$,
- globally for any initial condition if $0 < \alpha < \frac{4-2b}{N}$,
- globally if $\alpha = \frac{4-2b}{N}$, assuming $\|u_0\|_{L^2} < \|Q\|_{L^2}$,