Available online at www.sciencedirect.com
ScienceDirect
J. Differential Equations 262 (2017) 4232–4254
Journal of Differential Equations
www.elsevier.com/locate/jde

Symmetry and decay of traveling wave solutions to the Whitham equation

Gabriele Bruell *, Mats Ehrnström, Long Pei

Department of Mathematical Sciences, Norwegian University of Science and Technology, 7491 Trondheim, Norway

Received 26 August 2016; revised 15 December 2016
Available online 26 January 2017

Abstract

This paper is concerned with decay and symmetry properties of solitary-wave solutions to a nonlocal shallow-water wave model. An exponential decay result for supercritical solitary-wave solutions is given. Moreover, it is shown that all such solitary-wave solutions are symmetric and monotone on either side of the crest. The proof is based on the method of moving planes. Furthermore, a close relation between symmetric and traveling-wave solutions is established.

MSC: 35Q53; 35B06; 35B40; 35S30; 45K05

Keywords: Nonlocal equation; Solitary solutions; Symmetry; Exponential decay

1. Introduction

The dynamics of water waves for an inviscid perfect fluid are described by the Euler equations, complemented with suitable boundary conditions. Due to the intricate character of this system, a rigorous mathematical study of its solutions is challenging and it is one aim in the analysis of water waves to derive model equations which capture as many as possible of the phenomena

* Corresponding author.
E-mail addresses: gabriele.bruell@math.ntnu.no (G. Bruell), mats.ehrnstrom@math.ntnu.no (M. Ehrnström), long.pei@math.ntnu.no (L. Pei).

http://dx.doi.org/10.1016/j.jde.2017.01.011

displayed by water waves. In the context of irrotational, small-amplitude, shallow-water waves, it is well-known that the *Korteweg–de Vries equation* (KdV),

$$\eta_t + \frac{3}{2}\frac{c_0}{h_0}\eta\eta_x + c_0\eta_x + \frac{1}{6}c_0h_0^2\eta_{xxx} = 0, \tag{1.1}$$

can be rigorously deduced as a consistent approximation to the Euler equations [29]. Here, $\eta(t,x)$ describes the surface displacement from an undisturbed flow over a flat bottom at time $t \in [0,\infty)$ and spatial position $x \in \mathbb{R}$. The constant $c_0 := \sqrt{gh_0}$ is the limiting long-wave speed, h_0 is the undisturbed fluid depth and g denotes the gravitational constant of acceleration. Equation (1.1) may be equivalently expressed in nonlocal form as

$$\eta_t + \frac{3}{2}\frac{c_0}{h_0}\eta\eta_x + \mathcal{F}^{-1}\left(c(\xi)\right) * \eta_x = 0,$$

where $\mathcal{F}^{-1}$ denotes the inverse (spatial) Fourier transform, and

$$c(\xi) := c_0 - \frac{1}{6}c_0h_0^2\xi^2$$

is the dispersion relation of the KdV equation. Noticing that c is a second-order approximation of the exact dispersion relation of the linearized Euler equations,

$$m_{h_0}(\xi) := \left(\frac{g\tanh(\xi h_0)}{\xi}\right)^{\frac{1}{2}} = c_0 - \frac{1}{6}c_0h_0^2\xi^2 + O(\xi^4),$$

G.B. Whitham [36] suggested what is today termed the *Whitham equation*,

$$\eta_t + \frac{3}{2}\frac{c_0}{h_0}\eta\eta_x + \mathcal{F}^{-1}\left(m_{h_0}\right) * \eta_x = 0, \tag{1.2}$$

as an alternative to the KdV equation. Here, $K_{h_0} := \mathcal{F}^{-1}\left(m_{h_0}\right)$ is the integral kernel corresponding to a (genuinely) nonlocal Fourier multiplier operator with symbol m_{h_0}. This approach of *dispersion improving* is often applied to improve the modeling aspects of fluid dynamics equations [29], as it weakens the role of dispersion towards that of the full Euler equations. Equation (1.2) can also be obtained directly from the Euler equations via an exponential scaling [31]. From a consistency point of view, the equation (1.2) is neither a better nor a worse model than the KdV equation: their solutions both approximate shallow-water, small-amplitude gravity water-wave solutions of the Euler equations to the same order on appropriate time scales [29]. As described below, the Whitham equation (1.2) however has the property of capturing several of the mathematical *features* of the Euler equations, that the KdV equation does not (including nonlocality, break-down of solutions, modulational instability and highest waves).

The purpose of the present paper is to analyze geometric properties of solitary-wave solutions to the Whitham equation. We will show that the Whitham equation captures various characteristics of solitary solutions to the Euler equations. In the same physical setting as ours, it was