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Multiplicity of positive periodic solutions in the superlinear indefinite case via coincidence degree [☆]

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Abstract

We study the periodic boundary value problem associated with the second order nonlinear differential equation

$$u'' + cu' + \big(a^{+}(t) - \mu a^{-}(t)\big)g(u) = 0,$$

where $g(u)$ has superlinear growth at zero and at infinity, $a(t)$ is a periodic sign-changing weight, $c \in \mathbb{R}$ and $\mu > 0$ is a real parameter. Our model includes (for $c = 0$) the so-called nonlinear Hill's equation. We prove the existence of $2^m - 1$ positive solutions when $a(t)$ has m positive humps separated by m negative ones (in a periodicity interval) and μ is sufficiently large, thus giving a complete solution to a problem raised by G.J. Butler in 1976. The proof is based on Mawhin's coincidence degree defined in open (possibly unbounded) sets and applies also to Neumann boundary conditions. Our method also provides a topological approach to detect subharmonic solutions.

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1. Introduction

Let $\mathbb{R}^+ := [0, +\infty[$ denote the set of non-negative real numbers and let $g\colon \mathbb{R}^+ \to \mathbb{R}^+$ be a continuous function such that

$$g(0) = 0, \qquad g(s) > 0 \quad \text{for } s > 0. \tag{g_*}$$

In the present paper we study the problem of existence and multiplicity of positive T-periodic solutions to the second order nonlinear differential equation

$$u'' + cu' + w(t)g(u) = 0, \tag{1.1}$$

where $c \in \mathbb{R}$ and $w\colon \mathbb{R} \to \mathbb{R}$ is a T-periodic locally integrable weight function. Solutions to (1.1) are meant in the Carathéodory sense. A *positive solution* is a solution such that $u(t) > 0$ for all $t \in \mathbb{R}$. As is well known, solving the T-periodic problem for (1.1) is equivalent to find a solution of (1.1) satisfying the boundary conditions $u(0) = u(T)$, $u'(0) = u'(T)$, on $[0, T]$ (any other time-interval of length T can be equivalently chosen).

Our aim is to consider a nonlinear vector field $f(t, s) := w(t)g(s)$ satisfying suitable assumptions which cover the classical superlinear indefinite case, namely $g(s) = s^p$, with $p > 1$, and $w(t)$ a sign-changing coefficient. Our main result guarantees the existence of at least $2^m - 1$ positive T-periodic solutions provided that, in a time-interval of length T, the weight function presents m positive humps separated by negative ones and the negative parts of $w(t)$ are sufficiently large. To be more precise, it is convenient to express $w(t)$ as depending on a parameter $\mu > 0$ in this manner:

$$w(t) = a_\mu(t) := a^+(t) - \mu a^-(t),$$

where $a\colon \mathbb{R} \to \mathbb{R}$ is a T-periodic locally integrable function. As usual, we denote by

$$a^+(t) := \frac{a(t) + |a(t)|}{2} \quad \text{and} \quad a^-(t) := \frac{-a(t) + |a(t)|}{2}$$

the *positive part* and the *negative part* of $a(t)$, respectively. Then, a typical corollary of our main result (cf. Theorem 3.1) for equation

$$u'' + cu' + \big(a^+(t) - \mu a^-(t)\big)g(u) = 0 \tag{$\mathscr{E}$}$$

reads as follows.

Theorem 1.1. *Suppose that there exist $2m + 1$ points*

$$\sigma_1 < \tau_1 < \ldots < \sigma_i < \tau_i < \ldots < \sigma_m < \tau_m < \sigma_{m+1}, \quad \textit{with } \sigma_{m+1} - \sigma_1 = T,$$