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Harnack's inequality for a space–time fractional diffusion equation and applications to an inverse source problem

Junxiong Jia [a,b,*], Jigen Peng [a,b], Jiaqing Yang [a]

[a] *School of Mathematics and Statistics, Xi'an Jiaotong University, Xi'an 710049, China*

[b] *Beijing Center for Mathematics and Information Interdisciplinary Sciences (BCMIIS), China*

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Abstract

In this paper, we focus on a space–time fractional diffusion equation with the generalized Caputo's fractional derivative operator and a general space nonlocal operator (with the fractional Laplace operator as a special case). A weak Harnack's inequality has been established by using a special test function and some properties of the space nonlocal operator. Based on the weak Harnack's inequality, a strong maximum principle has been obtained which is an important characterization of fractional parabolic equations. With these tools, we establish a uniqueness result of an inverse source problem on the determination of the temporal component of the inhomogeneous term, which seems to be the first theoretical result of the inverse problem for such a general fractional diffusion model.

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Keywords: Inverse source problem; Fractional diffusion equation; Harnack's inequality; Strong maximum principle

* Corresponding author.

E-mail addresses: jjx323@mail.xjtu.edu.cn (J.X. Jia), jgpen@mail.xjtu.edu.cn (J. Peng), jiaq.yang@mail.xjtu.edu.cn (J. Yang).

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1. Introduction

Fractional partial differential equations grow up to be a popular research topic for its wide applications in physics [24], geological exploration [39] and so on. For mathematical properties, Baeumer, Meerschaert etc. [3,2,4] investigate time fractional parabolic equations from functional and probabilistic perspective. They construct stochastic solutions and discover a lot of interesting relations between fractional differential equations and stochastic process. By extending operator semigroup theory, mild solutions and some subordination principles have been obtained in [16,19] for some types of fractional parabolic equations. For time fractional parabolic equations, Zacher [34–37] constructs a series of theories concerned with weak solutions and Hölder continuities of the solutions. Recently, Allen, Caffarelli and Vasseur [1] obtain many important regularity properties for a space–time fractional parabolic equation. In this paper, we focus on a general fractional diffusion equation. Before going further, let us introduce some notations. For a real number $\gamma \in \mathbb{R}$, denote $g_\gamma(t)$ by

$$g_\gamma(t) = \frac{t^{\gamma-1}}{\Gamma(\gamma)}, \tag{1.1}$$

where $\Gamma(\cdot)$ represents the usual Gamma function. The notation $\partial_t^\alpha \cdot$ denotes the Riemann–Liouville fractional derivative defined by

$$\partial_t^\alpha f(t) := \frac{d}{dt}(g_{1-\alpha} * f(\cdot))(t), \tag{1.2}$$

where “$*$” denotes the usual convolution operator. The space–time nonlocal diffusion equation studied in this paper has the following form

$$\begin{cases} \partial_t^\alpha(u(x,t) - u_0(x)) + Lu(x,t) = f(x,t) & \text{in } \Omega \times [0,T], \\ u(x,t) = 0 & \text{in } \mathbb{R}^n \backslash \Omega,\ t \geq 0, \\ u(x,0) = u_0(x) & \text{in } \Omega, \text{ for } t = 0, \end{cases} \tag{1.3}$$

where $\alpha \in (0,1)$ and L is an integro-differential operator of the form

$$Lu(t,x) = \text{p.v.} \int_{\mathbb{R}^n} [u(t,x) - u(t,y)]k(x,y)dy. \tag{1.4}$$

The time-fractional operator used here could be called the generalized Caputo’s fractional derivative. For more details, we refer to [26]. The kernel $k : \mathbb{R}^n \times \mathbb{R}^n \to [0,\infty)$, $(x,y) \mapsto k(x,y)$ is assumed to be measurable with a certain singularity at the diagonal $x = y$.

Note that in the case $k(x,y) = c_{n,\beta}/|x-y|^{n+2\beta}$ with constant $c_{n,\beta} = \frac{\beta 2^{2\beta}\Gamma(\frac{n+2\beta}{2})}{\pi^{n/2}\Gamma(1-\beta)}$, the integral-differential operator L defined in (1.4) is equal to $(-\Delta)^\beta$ which is the pseudo-differential operator with symbol $|\xi|^{2\beta}$. Thus the operator L could be seen as a generalized fractional Laplace operator. And the following space–time fractional diffusion equation is a special case of equation (1.3)

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