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Stochastic homogenization of a front propagation problem with unbounded velocity

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Abstract

We study the homogenization of Hamilton–Jacobi equations which arise in front propagation problems in stationary ergodic media. Our results are obtained for fronts moving with possible unbounded velocity. We show, by an example, that the homogenized Hamiltonian, which always exists, may be unbounded. In this context, we show convergence results if we start with a compact initial front. On the other hand, if the media satisfies a *finite range of dependence* condition, we prove that the effective Hamiltonian is bounded and obtain classical homogenization in this context.

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1. Introduction

We study the homogenization, as $\varepsilon \to 0$, of the solution to the Hamilton–Jacobi equation

$$\begin{cases} u_t^\varepsilon = a\left(\frac{x}{\varepsilon}, \omega\right)|Du^\varepsilon| & \text{in} \quad \mathbb{R}^d \times (0,T) \\ u^\varepsilon(x,0,\omega) = u_0(x) & \text{in} \quad \mathbb{R}^d, \end{cases} \tag{1.1}$$

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where $a(y,\omega):\mathbb{R}^d\times\Omega\to\mathbb{R}$ is a stationary ergodic process and $(\Omega,\mathcal{F},\mathbb{P})$ is a probability space. Throughout the paper, we set $H(p,y,\omega)=a(y,\omega)|p|$, which stands for the Hamiltonian. It is well known (see [40] for example) that (1.1) is a level-set approach of fronts moving in the normal direction with oscillatory velocity $\mathcal{V}=a\left(\frac{x}{\varepsilon},\omega\right)$ (see also [15,43] and references therein).

Most known results in stochastic homogenization assume that the velocity $a(y,\omega)$ is bounded above and below by positive constants. This allows us to get uniform Lipschitz estimates for the solutions and then homogenization results for (1.1) as follows (see [44] for details): there exists $\overline{H}:\mathbb{R}^d\to\mathbb{R}$, called the effective Hamiltonian, such that

$$u^\varepsilon(x,t,\omega)\underset{\varepsilon\to 0}{\longrightarrow}\overline{u}(x,t),\quad \textit{almost surely}\text{ in }\omega, \tag{1.2}$$

and locally uniformly in (x,t), where $\overline{u}(x,t)$ is the unique solution of the deterministic Hamilton–Jacobi equation

$$\begin{cases}\overline{u}_t=\overline{H}(D\overline{u}) & \text{in}\quad \mathbb{R}^d\times(0,T)\\ \overline{u}(x,0)=u_0(x) & \text{in}\quad \mathbb{R}^d.\end{cases} \tag{1.3}$$

We work in the general setting where $a(y,\omega)$ is not assumed to be bounded above or below. Actually, we assume that $a(y,\omega)$ is positive and finite. The lack of a uniform lower bound was already studied in [9]. In our work, we concentrate on problem (1.1) with the lack of a uniform upper bound on the velocity $a(y,\omega)$. This means that the front can travel arbitrary fast. The main difficulty is that we have not uniform Lipschitz estimates for the solutions as in the classical cases. As a consequence, a new phenomenon appears. Indeed, the effective Hamiltonian, which always exists, can be equal to infinity, and then the limit problem (1.3) is ill-posed. In this context, we proved that, surprisingly, convergence results hold for the solution u^ε to (1.1) as $\varepsilon\to 0$, provided that we start with a compact initial front. On the other hand, assuming a finite range of dependence condition on the media, we established an upper bound for the effective Hamiltonian and proved classical homogenization for (1.1). The finite range of dependence condition roughly means that there is a length scale $l>0$ such that the random variables $a(y,\cdot)$ and $a(z,\cdot)$ are independent whenever $|y-z|\geq l$. This condition is not new in the framework of the stochastic homogenization of Hamilton–Jacobi equations, and has been already used in some works. For example, it has been used in [5,6] to obtain error estimates and rates of convergence for homogenization.

In what follows, we recall some homogenization results for Hamilton–Jacobi equations in both periodic and random cases. We emphasize that all these results are obtained for Hamiltonians $H(p,y,\omega)$ which are bounded from above uniformly in (y,ω). Periodic homogenization of coercive Hamiltonians was first studied by Lions, Papanicolaou and Varadhan [34] and, later, Evans [25,26]. Ishii established in [29] homogenization in almost periodic settings. It is worth to point out that we cannot extend techniques of periodic homogenization to the stochastic case. Indeed, Lions and Souganidis [35] proved that, in a general random media, the cell problem, which is the main tool in the study of the periodic homogenization, has no solution which is strictly sublinear at infinity. The stochastic homogenization of convex and coercive Hamiltonians was established independently by Souganidis [44] and Rezakhanlou and Tarver [41]. Results for viscous Hamilton–Jacobi equations were obtained by Lions and Souganidis [36] and Kosygina, Rezakhanlou and Varadhan [32], while problems with space-time oscillations were considered by Kosygina and Varadhan [33], Schwab [42] and Jing, Souganidis and Tran [30]. In [37] Lions and Souganidis proved homogenization results for the viscous case by using techniques of weak