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Exact singular behavior of positive solutions to nonlinear elliptic equations with a Hardy potential ☆

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Abstract

In this paper, we study the singular behavior at $x = 0$ of positive solutions to the equation

$$-\Delta u = \frac{\lambda}{|x|^2} u - |x|^\sigma u^p, \quad x \in \Omega\backslash\{0\},$$

where $\Omega \subset \mathbb{R}^N (N \geq 3)$ is a bounded domain with $0 \in \Omega$, and $p > 1$, $\sigma > -2$ are given constants. For the case $\lambda \leq (N-2)^2/4$, the singular behavior of all the positive solutions is completely classified in the recent paper [5]. Here we determine the exact singular behavior of all the positive solutions for the remaining case $\lambda > (N-2)^2/4$. In sharp contrast to the case $\lambda \leq (N-2)^2/4$, where several converging/blow-up rates of $u(x)$ are possible as $|x| \to 0$, we show that when $\lambda > (N-2)^2/4$, every positive solution $u(x)$ blows up in the same fashion:

$$\lim_{|x| \to 0} |x|^{\frac{2+\sigma}{p-1}} u(x) = \left[\lambda + \frac{2+\sigma}{p-1}\left(\frac{2+\sigma}{p-1} + 2 - N\right)\right]^{1/(p-1)}.$$

MSC: 35J55; 35B40

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Keywords: Hardy potential; Isolated singularity; Positive solution

1. Introduction

In this article, we investigate the singular behavior at $x=0$ of positive solutions to the equation

$$-\Delta u=\frac{\lambda}{|x|^2}u-|x|^\sigma u^p,\quad x\in\Omega\backslash\{0\}, \tag{1.1}$$

where $\Omega\subset\mathbb{R}^N (N\ge 3)$ is a bounded smooth domain with $0\in\Omega$, and $p>1$, $\sigma>-2$ are given constants. The right-hand side of equation (1.1) contains an inverse square term, which is usually called the Hardy potential.

The study of this kind of equations was motivated by the understanding of certain physical phenomena, such as the interaction among neutral atoms in Thomas–Fermi theory [1,2]. The rich phenomena exhibited by the behavior of the solution $u(x)$ near $x=0$ as the parameters p, σ and λ are varied have made (1.1) an interesting mathematical problem, attracting extensive investigations by many researchers. For example, when $\lambda=0$, the singular behavior of (1.1) (and its variations) is studied in [1–3,6,7,13,14]. The case $\lambda\ne 0$ but $\sigma=0$ is considered in [4,10], where the term u^p is replaced by a more general function $h(u)$ behaving like u^p; [10] mainly deals the case $\lambda\le\frac{(N-2)^2}{4}$, while [4] only considers the case $0<\lambda<\frac{(N-2)^2}{4}$. A complete classification of the behavior of all the positive solutions of (1.1) at $x=0$ for the case $\lambda\le\frac{(N-2)^2}{4}$ and $\sigma>-2$ is obtained recently in [5], where it even allows $|x|^\sigma$ to be replaced by a more general function $b(x)$ and u^p by a more general $h(u)$. For the special case (1.1), the results of [5] (see Corollaries 7.4 and 7.5 there) reveal the following rich behavior of the positive solutions of (1.1):

Theorem A. *Suppose that $-\infty<\lambda<(N-2)^2/4$ and u is any positive solution of (1.1). Define*

$$\tau:=\frac{N-2}{2}-\sqrt{\frac{(N-2)^2}{4}-\lambda},\ \ p^*:=1+\frac{2+\sigma}{N-2-\tau},\ \ p^{**}:=1+\frac{2+\sigma}{\tau},$$
$$\ell:=\lambda+\frac{2+\sigma}{p-1}\left(\frac{2+\sigma}{p-1}+2-N\right).$$

(1) If $1<p<p^*$, then as $|x|\to 0$, exactly one of the following holds:
α. $|x|^\tau u(x)$ converges to a positive constant;
β. $|x|^{N-2-\tau}u(x)$ converges to a positive constant;
γ. $|x|^{\frac{2+\sigma}{p-1}}u(x)$ converges to $\ell^{\frac{1}{p-1}}$.

(2) If $(p,\lambda)\in[p^*,+\infty)\times(-\infty,0]\cup[p^*,p^{**})\times(0,\frac{(N-2)^2}{4})$, then (1)$\alpha$ above holds.

(3) If $p=p^{**}$ and $0<\lambda<\frac{(N-2)^2}{4}$, then

$$\lim_{|x|\to 0}\left(|x|\big(\log\frac{1}{|x|}\big)^{1/(2+\sigma)}\right)^\tau u(x)=\left(\frac{N-2-2\tau}{p-1}\right)^{1/(p-1)}.$$

(4) If $p>p^{**}$ and $0<\lambda<\frac{(N-2)^2}{4}$, then (1)$\gamma$ above holds.

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