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Asymptotic blow-up analysis for singular Liouville type equations with applications

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Abstract

We generalize the pointwise estimates obtained in [2,19] and [34] concerning blow-up solutions of the Liouville type equation:

$$-\Delta u = |x|^{2\alpha} W(x)\, e^{u} \quad \text{in} \quad \Omega,$$

with $\Omega \subset \mathbb{R}^2$ open and bounded, $\alpha \in (-1, +\infty)$ and W any Lipschitz continuous function which satisfies $0 < a \le W \le b < \infty$. We focus to the case (left open in [2] and [34]) where the parameter $\alpha \in \mathbb{N}$, whose analysis is much more involved as we need to resolve the difficulty of a genuinely non radial behaviour of blow-up solutions. In the worst situation there is no chance (in general) to resolve the profile in the form of a solution of a Liouville equation in $\mathbb{R}^2$, instead we need to adopt iterated blow-up arguments.

Next, we refine our blow up analysis to cover a class of planar Liouville type problems (see (1.27)–(1.28) below) arising from the study of Cosmic Strings (cfr. [28,35]). In this context, we are able to distinguish between a single blow-up radial profile and the case of multiple blow-up profiles, typical of non radial solutions. As a consequence we obtain a (radial) symmetry result which is interesting in itself but also contributes towards the "sharp" solvability issue for the planar problem (1.27)–(1.28).

MSC: 35B45; 35J60; 35J99

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1. Introduction

The analysis of Liouville-type equations [24] with singular data [27], has attracted much attention in recent years due to their application to several issues of interest in Mathematics and Physics, such as Chern–Simons and Electroweak self-dual vortices [31,34,38], conformal metrics on surfaces with conical singularities [1,4,6,36], statistical mechanics of two-dimensional turbulence [10] and of self-gravitating systems [37] and cosmic strings [28,35], and more recently the theory of hyperelliptic curves [11].

Existence (and multiplicity) results for such class of equations (see for example [5,17,21,25] and references therein) are based on topological and variational methods that require in an essential way the compactness property of the solution set. To this purpose, refined blow up techniques have been developed for the description of the "local" asymptotic profile of solution sequences around a (blow-up) point, where "bubbling" phenomena develop, see [8,9,19,20], and more recently [2,16,32,33,39].

In this note, we shall contribute in this direction and provide pointwise estimates for the profile of a solution sequence in the situation where "multiple-bubbles" occur. To be more precise, let us mention that, after introducing suitable coordinates, one can "localize" the problem around the origin and so analyse the behaviour of a solution sequence $u_n \in C^0(\overline{B_r})$ satisfying:

$$\begin{cases} -\Delta u_n = |x|^{2\alpha_n} W_n(x)\, e^{u_n} \ \text{ in } \ B_r, \\ |u_n(x) - u_n(y)| \leq c_0, \ \text{ for } |x| = |y| = r, \\ \int\limits_{B_r} |x|^{2\alpha_n} W_n(x)\, e^{u_n} \leq c_1, \end{cases} \tag{1.1}$$

with suitable constants $c_0 \geq 0$, $c_1 > 0$ and $B_r := \{x \in \mathbb{R}^2 : |x| < r\}$, $r > 0$.

We shall assume that:

$$0 < a \leq \min_{\overline{B_r}} W_n \leq \max_{\overline{B_r}} W_n \leq b < +\infty, \quad \max_{\overline{B_r}} \|\nabla W_n\| \leq A, \tag{1.2}$$

and let

$$\alpha_n \to \alpha \in (-1, +\infty), \ \text{ as } \ n \to +\infty, \tag{1.3}$$

and (by taking r smaller if necessary) we also suppose that u_n admits the origin as its only blow up point in B_r. In other words, $\forall\, 0 < \varepsilon < r$ there exist $C_\varepsilon > 0$ (depending on ε) such that:

$$\max_{\overline{B_r}} u_n \to +\infty, \quad \limsup_{n\to+\infty} \left(\max_{\overline{B_r} \setminus B_\varepsilon} u_n \right) \leq C_\varepsilon. \tag{1.4}$$

By well known results, we know the following:

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