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A global 2D well-posedness result on the order tensor liquid crystal theory

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Abstract

In [18] Paicu and Zarnescu have studied an order tensor system which describes the flow of a liquid crystal. They have proven the existence of weak solutions, the propagation of higher regularity, namely H^s with $s > 1$ and the weak-strong uniqueness in dimension two. This paper is devoted to fill the gap of their results, namely to propagate the low regularity, namely H^s for $0 < s < 1$ and to prove the uniqueness of the weak solutions. For the completeness of this research, we also propose an alternative approach in order to prove the existence of weak solutions.

MSC: 76A15; 35Q30; 35Q35; 76D03

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1. Introduction and main results

The theory of liquid crystal materials has attracted much attention over the recent decades. Generally, the physical state of a material can be determined by the motion degree of freedom about its molecules. Certainly, the widespread physical states of matter are the solid, the liquid and the gas ones. If the movement degree of freedom is almost zero, namely the forces which act on the molecules don't allow any kind of movement, forcing the material structure to be confined

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in a specific order, then we are classifying a solid material. If such degree still preserves a strong intermolecular force but it is not able to restrict the molecules to lie on a regular organization, then we are considering a fluid state of matter. Finally in the gas phase the forces and the distance between the molecules are weak and large respectively, so that the material is not confined and it is able to extend its volume.

However, some materials possess some common liquid features as well as some solid properties, namely the liquid crystals. As the name suggests, a liquid crystal is a compound of fluid molecules, which has a state of matter between the ordinary liquid one and the crystal solid one. The molecules have not a positional order but they assume an orientation which can be modified by the velocity flow. At the same time a variation of the alignment can induce a velocity field as well. In a common liquid (more correctly an isotropic liquid) if we consider the orientation of a single molecule then we should see the random variation of its position. Nevertheless, in a crystal liquid, we see an amount of orientational order.

It is well-documented that liquid crystals have been well-known for more than a century, however they have received a growth in popularity and much study only in recent decades, since they have attracted much attention thanks to their potential applications (see for instance [3]).

Commonly, in literature the liquid crystals are categorized by three sub-families, namely the nematics, the cholesterics and the smectics. On a nematic liquid crystal, the molecules have the same alignment with a preferred direction, however their positions are not correlated. On a cholesteric liquid crystal we have a foliation of the material where on each plaque the molecules orient themselves with the same direction (which could vary moving on the foliation). As in the nematic case, a cholesteric liquid crystal doesn't require any kind of relation between the positions of the molecules. At last, on a smectic liquid crystal we have still a privileged direction for all the molecules, as in the nematic case, however the position of them is bonded by a stratification. In addition to the orientational ordering, the molecules lie in layers.

1.1. The order tensor theory

A first mathematical approach to model the generic liquid crystals has been proposed by Ericksen [5] and Leslie [10] over the period of 1958 through 1968. Even if they have presented a system which has been extensively studied in literature, for instance in [12] and [22], several mathematical challenges and difficulties reside in such model. Hence, in 1994, Baris and Edwards [2] proposed an alternative approach based on the concept of order Q-tensor, that one can find also in physical literature, for example [4] and [20]. The reader can find an exhaustive introduction to the Q-tensor theory in a recent paper of Mottram and Newton [16], however we present here some hints in order to introduce the Q-tensor system.

Let us assume that our material lies on a domain Ω of $\mathbb{R}^3$. A first natural strategy to model the molecules orientation is to introduce a vector field n, the so called director field (see for instance [13]), which returns value on $\mathbb{S}^2$, the boundary of the unit sphere on $\mathbb{R}^3$. Here $n(t,x)$ is a specific vector for any fixed time and for any $x \in \Omega$. An alternative approach is not to consider a precise position on $\mathbb{S}^2$ but to establish the probability that $n(t,x)$ belongs to some measurable subset $\mathcal{A} \subseteq \mathbb{S}^2$. Therefore we introduce a continuously distributed measure $\mathcal{P}$ on $\mathbb{S}^2$, driven by a density ρ

$$\mathcal{P}(\mathcal{A}) = \int_{\mathcal{A}} \rho(P)\mathrm{d}\sigma(P) = \int_{\mathcal{A}} \mathrm{d}\rho(P).$$

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