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Global smooth solution of a two-dimensional nonlinear singular system of differential equations arising from geostrophics

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Abstract

Consider the Cauchy problems for the following two-dimensional nonlinear singular system of differential equations arising from geostrophics

$$\frac{\partial}{\partial t}[\gamma(\psi_1-\psi_2)-\triangle\psi_1]+\alpha(-\triangle)^\rho\psi_1+\beta\frac{\partial\psi_1}{\partial x}+J(\psi_1,\gamma(\psi_1-\psi_2)-\triangle\psi_1)=0,$$
$$\frac{\partial}{\partial t}[\gamma\delta(\psi_2-\psi_1)-\triangle\psi_2]+\alpha(-\triangle)^\rho\psi_2+\beta\frac{\partial\psi_2}{\partial x}+J(\psi_2,\gamma\delta(\psi_2-\psi_1)-\triangle\psi_2)=0,$$
$$\psi_1(x,y,0)=\psi_{01}(x,y),\qquad \psi_2(x,y,0)=\psi_{02}(x,y).$$

In this system, $\alpha>0$, $\gamma>0$, $\delta>0$ and $\rho>0$ are positive constants, $\beta\neq 0$ is a real nonzero constant, the Jacobian determinant is defined by $J(p,q)=\frac{\partial p}{\partial x}\frac{\partial q}{\partial y}-\frac{\partial p}{\partial y}\frac{\partial q}{\partial x}$.

The existence and uniqueness of the global smooth solution of the system of differential equations are very important in applied mathematics and geostrophics, but they have been open for a long time. The

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singularity generated by the linear parts, the strong couplings of the nonlinear functions and the fractional order of the derivatives make the existence and uniqueness very difficult to study.

Very recently, we found that there exist a few special structures in the system. The main purpose of this paper is to couple together the special structures and an unusual method for establishing the uniform energy estimates to overcome the main difficulty to accomplish the existence and uniqueness of the global smooth solution: $\psi_1 \in C^\infty(\mathbb{R}^2 \times \mathbb{R}^+)$ and $\psi_2 \in C^\infty(\mathbb{R}^2 \times \mathbb{R}^+)$, for all $\rho > 3/2$. The new energy method enables us to make complete use of the special structures of the nonlinear singular system. The results obtained in this paper provide positive solutions to very important open problems and greatly improve many previous results about nonlinear singular systems of differential equations arising from geostrophics.

MSC: 35Q20

Keywords: Cauchy problems; Systems of differential equations; Special structures; Global smooth solution; Existence and uniqueness; Uniform energy estimates; Leray–Schauder's fixed point principle

1. Introduction

1.1. The mathematical model equations

Consider the Cauchy problems for the following two-dimensional nonlinear singular system of differential equations arising from geostrophics

$$\frac{\partial}{\partial t}[\gamma(\psi_1 - \psi_2) - \triangle\psi_1] + \alpha(-\triangle)^\rho \psi_1 + \beta\frac{\partial \psi_1}{\partial x} + J(\psi_1, \gamma(\psi_1 - \psi_2) - \triangle\psi_1) = 0, \quad (1)$$

$$\frac{\partial}{\partial t}[\gamma\delta(\psi_2 - \psi_1) - \triangle\psi_2] + \alpha(-\triangle)^\rho \psi_2 + \beta\frac{\partial \psi_2}{\partial x} + J(\psi_2, \gamma\delta(\psi_2 - \psi_1) - \triangle\psi_2) = 0, \quad (2)$$

$$\psi_1(x, y, 0) = \psi_{01}(x, y), \qquad \psi_2(x, y, 0) = \psi_{02}(x, y). \quad (3)$$

Also consider the Cauchy problems for the following linear singular system of differential equations

$$\frac{\partial}{\partial t}[\gamma(\phi_1 - \phi_2) - \triangle\phi_1] + \alpha(-\triangle)^\rho \phi_1 + \beta\frac{\partial \phi_1}{\partial x} = 0, \quad (4)$$

$$\frac{\partial}{\partial t}[\gamma\delta(\phi_2 - \phi_1) - \triangle\phi_2] + \alpha(-\triangle)^\rho \phi_2 + \beta\frac{\partial \phi_2}{\partial x} = 0, \quad (5)$$

$$\phi_1(x, y, 0) = \phi_{01}(x, y), \qquad \phi_2(x, y, 0) = \phi_{02}(x, y). \quad (6)$$

In these systems of differential equations, $\alpha > 0$, $\gamma > 0$, $\delta > 0$ and $\rho > 0$ are positive constants, $\beta \neq 0$ is a real nonzero constant, $\frac{1}{\alpha}$ represents the Reynolds number, $\psi_1 = \psi_1(x, y, t)$ and $\psi_2 = \psi_2(x, y, t)$ represent stream functions of top layer and bottom layer in convective fluids. The constant $\delta = D_2/D_1 \in (0, 1)$, where $D_1 > 0$ and $D_2 > 0$ represent the depths of the top layer and the bottom layer, respectively. Moreover, the Jacobian determinant is defined by

$$J(p, q) = \frac{\partial p}{\partial x}\frac{\partial q}{\partial y} - \frac{\partial p}{\partial y}\frac{\partial q}{\partial x}, \quad (7)$$