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Locally bounded global solutions to a chemotaxis consumption model with singular sensitivity and nonlinear diffusion

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Abstract

We show the existence of locally bounded global solutions to the chemotaxis system

$$\begin{cases} u_t = \nabla\cdot(D(u)\nabla u) - \nabla\cdot(\frac{u}{v}\nabla v) & \text{in } \Omega\times(0,\infty)\\ v_t = \Delta v - uv & \text{in } \Omega\times(0,\infty)\\ \partial_\nu u = \partial_\nu v = 0 & \text{in } \partial\Omega\times(0,\infty)\\ u(\cdot,0)=u_0,\ v(\cdot,0)=v_0 & \text{in } \Omega \end{cases}$$

in smooth bounded domains $\Omega\subset\mathbb{R}^N$, $N\ge 2$, for $D(u)\ge\delta u^{m-1}$ with some $\delta>0$, provided that $m>1+\frac{N}{4}$.

MSC: 35K55; 35A01; 35K65; 92C17

Keywords: Keller–Segel; Chemotaxis; Nonlinear diffusion; Global existence; Boundedness

1. Introduction

Even simple, small organisms can exhibit comparatively complex and macroscopically apparent collective behaviour. Bacteria of the species *E. coli*, for example, when set in a capillary

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tube featuring a gradient of nutrient concentration form bands that are visible to the naked eye and migrate with constant speed. Following experimental works of Adler (see e.g. [1,2]), in 1971 Keller and Segel [14] introduced a phenomenological model to capture this kind of behaviour, a prototypical version of which is given by

$$\begin{cases} u_t = \nabla \cdot (D(u)\nabla u) - \nabla \cdot (\frac{u}{v}\nabla v) & \text{in } \Omega \times (0,\infty) \\ v_t = \Delta v - uv & \text{in } \Omega \times (0,\infty) \end{cases} \tag{1}$$

with $D(u) \equiv 1$. Herein, u represents the density of bacteria and v is used to denote the concentration of the nutrient. In the model in [14], the diffusion coefficient $D(u)$ is supposed to be constant, thus leading to the typical effect of linear diffusion which causes any population to spread with infinite speed of propagation. In order to avoid this (biologically clearly unrealistic) behaviour, it might be desirable to allow for diffusion of porous medium type (i.e. $D(u) = u^{m-1}$), cf. also [3, p. 1665].

Nevertheless, starting with [14], the model with linear diffusion has successfully been employed to find travelling wave solutions (see e.g. the overview in [37] and references cited therein) and also their stability has been investigated [19,27].

In spite of the rich literature concerned with travelling wave solutions (for such solutions to related systems see also [24,25,20], or [11,26]), little is known about existence of solutions for more general initial data (see below).

The difficulty lies in the hazardous combination of the consumptive effect of the second equation on the nutrient concentration with the singular chemotactic sensitivity in the first: While the second equation compels v to shrink, it is the cross-diffusive contribution of the chemotaxis term that seeks to enlarge the solutions to (1). And it is this very term that is furnished with a large coefficient whenever v becomes small.

For a moment leaving aside the logarithmic shape of the sensitivity in $\nabla \cdot (\frac{u}{v}\nabla v) = \nabla \cdot (u\nabla \log v)$, we are led to the system

$$\begin{cases} u_t = \Delta u - \nabla \cdot (u\nabla v), \\ v_t = \Delta v - uv, \end{cases} \tag{2}$$

which also appears as part of chemotaxis fluid systems intensively studied during the past six years. (The interested reader can consult the introduction of [16].) Even in (2), global existence of classical solutions is not yet known, apart from 2-dimensional settings [41] or under smallness conditions on v_0 [35].

Although the mathematical difficulty in treating the system vastly increases when a logarithmic sensitivity is included, this form is important. Not only is it needed for the emergence of travelling waves [14,13,32], there are also models giving a detailed mechanistic basis [45] and experimental evidence asserting this form [12].

In those Keller–Segel models (cf. [10,9,3]) where v does not stand for a nutrient to be consumed but a signalling substance produced by the bacteria themselves, i.e. the evolution is governed by

$$\begin{cases} u_t = \Delta u - \chi\nabla \cdot (\frac{u}{v}\nabla v), \\ v_t = \Delta v - v + u, \end{cases}$$