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A rigorous justification of the Matthews–Cox approximation for the Nikolaevskiy equation

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Abstract

The Nikolaevskiy equation is an example of a pattern forming system with marginally stable long modes. It has the unusual property that the typical Ginzburg–Landau scaling ansatz for the description of propagating patterns does not yield asymptotically consistent amplitude equations. Instead, another scaling proposed by Matthews and Cox can be used to formally derive a consistent system of modulation equations. We give a rigorous proof that this system makes correct predictions about the dynamics of the Nikolaevskiy equation.

Keywords: Amplitude equations; Multiscale analysis; Approximation; Pattern formation

1. Introduction

The Nikolaevskiy partial differential equation, given by

$$\partial_t u + u\partial_x u = -\partial_x^2 \left[r - (1+\partial_x^2)^2 \right] u,$$

($x \in \mathbb{R}$, $t \geq 0$, $u(x,t) \in \mathbb{R}$) was proposed as a one-dimensional model for seismic waves in viscoelastic media, see [1]. It also serves as a paradigmatic model for a pattern forming system with Galilean invariance, see [4]. For our multiscale analysis near the onset of pattern formation, i.e., in the case $0 < r \ll 1$, it is convenient to introduce a small parameter $\varepsilon > 0$, such that $r = \varepsilon^2$, and write the Nikolaevskiy equation as

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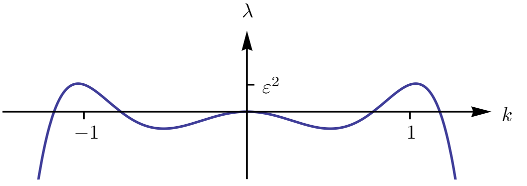

Fig. 1. Linear dispersion relation for the Nikolaevskiy equation: Turing instability with marginally stable long modes.

$$\partial_t u = L_\varepsilon u - \frac{1}{2}\partial_x(u^2), \qquad \text{where} \qquad L_\varepsilon = \partial_x^2(1+\partial_x^2)^2 - \varepsilon^2\partial_x^2. \tag{1}$$

Looking at the linear dispersion relation,

$$\lambda = -k^2(1-k^2)^2 + \varepsilon^2 k^2,$$

for modes $u(x,t) = \mathrm{e}^{\mathrm{i}kx+\lambda t}$, we see that for $\varepsilon > 0$ the spatially homogeneous steady state $u = 0$ becomes linearly unstable via a short wave instability. In addition to the classical Turing instability we also have a curve of eigenvalues touching the imaginary axis at the wave number $k = 0$, see Fig. 1. Hence, we have a spectral situation as considered in [3,9]. There, we derived amplitude equations for the propagation of small spatially periodic patterns using the typical Ginzburg–Landau scaling $X = \varepsilon x$, $T = \varepsilon^2 t$ for the large spatial and temporal scale, respectively, an $\mathcal{O}(\varepsilon)$ amplitude scaling for the pattern modes and an $\mathcal{O}(\varepsilon^2)$ amplitude scaling for the long modes.

In [4], Matthews and Cox pointed out that for the Nikolaevskiy equation such a scaling leads to amplitude equations that are asymptotically inconsistent in the sense that they contain $\mathcal{O}(1/\varepsilon)$ coefficients. Instead, they proposed an $\mathcal{O}(\varepsilon^{3/2})$ amplitude scaling of the pattern mode. Using the ansatz

$$\varepsilon^{3/2}\psi_{MC}(x,t) = \varepsilon^{3/2}A_1(\varepsilon x, \varepsilon^2 t)\mathrm{e}^{\mathrm{i}x} + \text{c.c.} + \varepsilon^2 A_0(\varepsilon x, \varepsilon^2 t),$$

where “c.c.” denotes the complex conjugate of the terms to the left, they derived the following system of amplitude equations for (1):

$$\begin{aligned} \partial_T A_1 &= 4\partial_X^2 A_1 + A_1 - \mathrm{i}A_1A_0, \\ \partial_T A_0 &= \partial_X^2 A_0 - \partial_X(|A_1|^2). \end{aligned} \tag{2}$$

While it is reasonable to assume that $\varepsilon^{3/2}\psi_{MC}$ with A_1 and A_0 given as solutions of (2) is a good approximation to a true solution of (1), it is not obvious. In fact, there are cases where approximations based on formally correctly derived amplitude equations make wrong predictions about the original system, see, e.g., [6–8].

In case of the Nikolaevskiy equation, so far, the question of validity has been tackled by numerical investigations only. While in [4,10] the simulations seem to verify the unusual scaling by Matthews and Cox, more recent results raise doubts, see [11].

In this paper we give a rigorous proof that the Matthews–Cox approximation is indeed valid and that all the dynamics of the Matthews–Cox system (2) in the respective phase spaces can be found in the Nikolaevskiy equation as well. For the proof of validity we apply methods that have already proven useful in the context of the justification of the Ginzburg–Landau approximation.