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Restricting Riesz–Morrey–Hardy potentials [☆]

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Abstract

To strengthen the weak restriction principle discovered in both [2, Theorem 5.1] for $\lambda = \kappa$ and [32, Lemma 2.1] for $\lambda < \kappa$, this paper shows that the Riesz operator I_α of order α maps the Morrey–Hardy space $LH^{p,\kappa}$ to the Morrey–Radon space $L_\mu^{q,\lambda}$ continuously if and only if

$$\sup_{(x,r)\in\mathbb{R}^n\times(0,\infty)} \mu\big(B(x,r)\big)/r^\beta < \infty \text{ provided } \begin{cases} 0 < \lambda \le \kappa \le n; \\ 0 < \alpha < n; \\ 1 \le p < \kappa/\alpha; \\ n - \alpha p < \beta \le n; \\ 0 < q = p(\beta + \lambda - n)/(\kappa - \alpha p). \end{cases}$$

Moreover, this brand-new restriction principle is optimal in the sense that if $n = \lambda > \kappa > 0$ then the iff-statement fails. Quite remarkably, such an optimization confirms affirmatively the conjecture in [7, Remark 3.3]: if

$$\alpha < \alpha p < \kappa < \lambda = n \quad \& \quad q = p\beta/(\kappa - \alpha p)$$

then there is no continuity of $I_\alpha : L^{p,\kappa} \to L_\mu^{q,n}$ for an arbitrary Radon measure μ with $|||\mu|||_\beta < \infty$.

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1. Introduction

A weak restriction theorem for Riesz–Morrey potentials due to Adams (see [2, Theorem 5.1] for $\lambda = \kappa$) and Xiao (see [32, Lemma 2.1] for $\lambda < \kappa$) states that if

$$\begin{cases} 0 < \lambda \le \kappa \le n; \\ 0 < \alpha < n; \\ 1 \le p < \kappa/\alpha; \\ n - \alpha p < \beta \le n; \\ 0 < q = p(\beta + \lambda - n)/(\kappa - \alpha p), \end{cases}$$

then the Riesz operator I_α, as a non-trivial extension of the Newtonian or electrostatic potential in universal gravitation, continuously maps the Morrey space $L^{p,\kappa}$ to the weak Morrey–Radon space $L^{q,\lambda}_{\mu,\infty}$ when and only when

$$|||\mu|||_\beta \equiv \sup_{(x,r)\in\mathbb{R}^n\times(0,\infty)} r^{-\beta}\mu\big(B(x,r)\big) < \infty.$$

Here and henceforth, $B(x,r)$ is the $1 \le n$-dimensional Euclidean open ball with center $x \in \mathbb{R}^n$ and radius $r \in (0,\infty)$. Also, for $(q,\lambda) \in [1,\infty) \times (0,n]$ and a (nonnegative) Radon measure μ on $\mathbb{R}^n$, the weak Morrey–Radon space $L^{q,\lambda}_{\mu,\infty}$ consists of all μ-measurable functions f with

$$\|f\|_{L^{q,\lambda}_{\mu,\infty}} \equiv \sup_{(x,r,t)\in\mathbb{R}^n\times(0,\infty)\times(0,\infty)} \Big(r^{\lambda-n}t^q\mu\big(\{y \in B(x,r) : |f(y)| > t\}\big)\Big)^{\frac{1}{q}} < \infty.$$

Naturally, as a subspace of $L^{q,\lambda}_{\mu,\infty}$, the symbol $L^{q,\lambda}_{\mu}$ expresses the Morrey–Radon space of all μ-measurable functions f in $\mathbb{R}^n$ with

$$\|f\|_{L^{q,\lambda}_{\mu}} \equiv \sup_{(x,r)\in\mathbb{R}^n\times(0,\infty)} \left(r^{\lambda-n}\int_{B(x,r)} |f|^q\,d\mu\right)^{\frac{1}{q}} < \infty.$$

Especially, if μ is just the n-dimensional Lebesgue measure ν and

$$(q,\lambda) = (p,\kappa) \in [1,\infty) \times (0,n)$$

then $L^{q,\lambda}_{\mu}$ will be simply written as the classical Morrey space $L^{p,\kappa}$ whose specialty $L^{p,n}$ coincides with the Lebesgue space L^p over $\mathbb{R}^n$. Moreover, given $\alpha \in (0,n)$ the Riesz operator I_α acting on a Lebesgue ν-measurable function f in $\mathbb{R}^n$ is determined by