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Bifurcation diagrams for Hamiltonian nilpotent centers of linear plus cubic homogeneous polynomial vector fields

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Abstract

Following the work done in [8] we provide the bifurcation diagrams for the global phase portraits in the Poincaré disk of all Hamiltonian nilpotent centers of linear plus cubic homogeneous planar polynomial vector fields.

MSC: primary 34C07, 34C08

Keywords: Hamiltonian nilpotent center; Cubic polynomial; Vector fields; Phase portrait; Bifurcation diagram

1. Introduction and statement of the main results

To distinguish when a singular point of a real planar polynomial differential system is a focus or a center is one of the main problems in the qualitative theory of differential systems. The definition of a *center* mainly goes back to Poincaré, who in [18] defines a *center* for a vector field on the real plane as a singular point having a neighborhood filled with periodic orbits with the exception of the singular point.

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There are three types of centers for analytic differential systems. An analytic system having a center can be written in one of the following forms after an affine change of variables and a rescaling of the time variable:

$\dot{x} = -y + P(x, y)$, $\dot{y} = x + Q(x, y)$, called a *linear type center*,
$\dot{x} = y + P(x, y)$, $\dot{y} = Q(x, y)$, called a *nilpotent center*,
$\dot{x} = P(x, y)$, $\dot{y} = Q(x, y)$, called a *degenerate center*,

where $P(x, y)$ and $Q(x, y)$ are real analytic functions without constant and linear terms, defined in a neighborhood of the origin. Poincaré [19,20] and Lyapunov [15] provide an algorithm for the characterization of linear type centers, see also Chazy [5] and Moussu [17] for other characterizations of the linear type centers. There is also an algorithm for the characterization of nilpotent and some classes of degenerate centers due to Chavarriga et al. [4], Cima and Llibre [6], Giacomini et al. [11], and Giné and Llibre [12].

The study of centers of polynomial differential systems started with Dulac for complex systems (see [9]), and after that it came the characterization of the centers of the real quadratic polynomial differential systems, and these studies are historically traced back to mainly Kapteyn [13,14] and Bautin [1]. For more recent works see Schlomiuk [21] and Żołądek [26]. Even though the centers of polynomial differential systems with degrees higher than 2 are not classified completely, there are many partial results. For instance the linear type centers of cubic polynomial differential systems of the form linear with homogeneous nonlinearities of degree 3 were characterized by Malkin [16], and by Vulpe and Sibiirski [24]. On the other hand, for systems with higher degree homogeneous nonlinearities the linear type centers are not fully characterized, but see Chavarriga and Giné [2,3] for some of the main results. Despite these advances the path to characterize and classify the centers of all polynomial differential systems of degree 3 and greater is long. We note that there are some interesting results in some subclasses of cubic systems due to the works of Rousseau and Schlomiuk [22], and Żołądek [27,28].

In [23] Vulpe provides all the global phase portraits of quadratic polynomial differential systems having a center. Then the bifurcation diagrams for the global phase portraits of these systems is given in [21]. The global phase portraits of linear type and nilpotent centers of polynomial differential systems having linear plus cubic homogeneous terms are presented in [7] (see also [10]) and in [8] respectively. In this work we provide the bifurcation diagrams for the global phase portraits of the latter.

Here, in this paper, we say that two vector fields on the Poincaré disk are *topologically equivalent* if there exists a homeomorphism from one onto the other which sends orbits to orbits preserving or reversing the direction of the flow. In [8] the global phase portraits on the Poincaré disk of all Hamiltonian planar polynomial vector fields with only linear and cubic homogeneous terms having a nilpotent center at the origin are given by the following theorem:

Theorem 1. *A Hamiltonian planar polynomial vector field with linear plus cubic homogeneous terms has a nilpotent center at the origin if and only if, after a linear change of variables and a rescaling of its independent variable, it can be written as one of the following six classes:*

(I) $\dot{x} = ax + by$, $\dot{y} = -\dfrac{a^2}{b}x - ay + x^3$, *with* $b < 0$.

(II) $\dot{x} = ax + by - x^3$, $\dot{y} = -\dfrac{a^2}{b}x - ay + 3x^2y$, *with* $a > 0$ *and* $b \neq 0$.