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Convexity of level sets of minimal graph on space form with nonnegative curvature

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Abstract

For the minimal graph defined on a convex ring in the space form with nonnegative curvature, we obtain the regularity and the strict convexity about its level sets by the continuity method.

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Keywords: Level sets; Constant rank theorem; Minimal graph

1. Introduction

The Geometry especially the convexity of level sets of the solutions to elliptic partial differential equations has been interesting to us for a long time. For instance, Alfhors [1] concluded that level curves of Green function on simply connected convex domain in the plane are the convex Jordan curves. Shiffman [31] studied the minimal annulus in R^3 whose boundary consists of two closed convex curves in parallel planes P_1, P_2, he derived that the intersection of the surface with any parallel plane P, between P_1 and P_2, is a convex Jordan curve. In 1957, Gabriel [11] proved that the level sets of the Green function on a 3-dimensional bounded convex domain are strictly convex and Lewis [21] extended Gabriel's result to p-harmonic functions in higher dimensions. Makar-Limanov [24] and Brascamp–Lieb [4] got the results on the Poisson equation

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and first eigenvalue equation with Dirichlet boundary value problem on bounded convex domain. Caffarelli–Spruck [7] generalized Lewis's results [21] to a class of semilinear elliptic partial differential equations. Motivated by the result of Caffarelli–Friedman [5], Korevaar [19] gave a new proof on the results of Gabriel and Lewis [11,21] using the deformation process and the constant rank theorem of the second fundamental form of convex level sets of p-harmonic function. Moreover, he also concluded in his paper [19] that level sets of minimal graph defined on convex rings are strictly convex. Kawohl [18] gave a survey of this subject. For more recent related extensions, please see the papers by Bianchin–Longinetti–Salani [3], Xu [38] and Bian–Guan–Ma–Xu [2].

On the curvature estimates of the level sets, Ortel–Schneider [29], Longinetti [22,23] proved that the curvature of level curves attains its minimum on the boundary (see also Talenti [33] for related results) for 2-dimensional harmonic function with convex level curves. Furthermore, Longinetti studied the precise relation between the curvature of the convex level lines and the height of minimal graph in [23]. The curvature estimate of the level sets of the solution to partial differential equations then had no new progress until recently, Ma–Ou–Zhang [25] got the Gaussian curvature estimates of the convex level sets of harmonic functions which depend on the Gaussian curvature of the boundary and the norm of the gradient on the boundary in R^n. Furthermore, in [26] the concavity of the Gaussian curvature of the convex level sets of p-harmonic functions with respect to the height was derived to describe the variation of the curvature along the height of the function. In [16], the lower bound of the principal curvature of the convex level sets of the solution to a kind of fully nonlinear elliptic equations was derived. For Poisson equations and a class of semilinear elliptic partial differential equations, Caffarelli–Spruck [7] concluded that the level sets of their solutions are all convex with respect to the gradient direction, the curvature estimate of the level sets has been got by Wang–Zhang [37], and in the same paper they also described the geometrical properties of the level sets of the minimal graph. In the sequel, following the technique in [26], Wang [34] got the precise relation between the curvature of the convex level sets and the height of minimal graph of general dimensions which generalized the previous results of Longinetti [23]. In [35] and [36], similar results about the geometrical properties of the solution to some partial differential equations were derived.

For the Riemannian manifold case, Papadimitrakis [30] concluded the convexity of the level curves of harmonic functions on convex rings in the hyperbolic plane via one complex variable tools. Ma–Zhang [27] generalized Papadimitrakis's results to space form of general dimensions. Partial results in [27] can be stated as follows.

Theorem 1.1. *([27]) Let (M^n, g) be a space form with constant sectional curvature 1 or -1, and Ω_0 and Ω_1 be bounded smooth strictly convex domains in M^n, $n \geq 2$ and $\bar{\Omega}_1 \subseteq \Omega_0$. Let ω satisfy*

$$\begin{cases} \triangle\omega = 0 & \text{in} \quad \Omega = \Omega_0 \setminus \bar{\Omega}_1, \\ \omega = 0 & \text{on} \quad \partial\Omega_0, \\ \omega = 1 & \text{on} \quad \partial\Omega_1. \end{cases}$$

Then $\nabla\omega \neq 0$ is valid everywhere in Ω and all the level sets of ω are strictly convex with respect to $\nabla\omega$.

Based on the above strict convexity of the level sets of harmonic functions defined on the convex ring in space forms, following the technique in [19], we come in this paper to consider another important geometrical object, the minimal graph defined on the convex ring in space form with nonnegative curvature. We mainly get the following theorem.