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Inverse acoustic scattering problem in half-space with anisotropic random impedance

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Abstract

We study an inverse acoustic scattering problem in half-space with a probabilistic impedance boundary value condition. The Robin coefficient (surface impedance) is assumed to be a Gaussian random function with a pseudodifferential operator describing the covariance. We measure the amplitude of the backscattered field averaged over the frequency band and assume that the data is generated by a single realization of λ. Our main result is to show that under certain conditions the principal symbol of the covariance operator of λ is uniquely determined. Most importantly, no approximations are needed and we can solve the full non-linear inverse problem. We concentrate on anisotropic models for the principal symbol, which leads to the analysis of a novel anisotropic spherical Radon transform and its invertibility.

1. Introduction

In this work we study inverse acoustic scattering in half-space. We assume that the time-harmonic acoustic field u satisfies the Helmholtz equation

$$(\Delta + k^2)u(x) = \delta_y(x), \quad x = (x_1, x_2, x_3) \in \mathbb{R}^3_+, \tag{1}$$

where $\mathbb{R}^3_+ = \mathbb{R}^2 \times (0, \infty)$, $k \in \mathbb{R}_+$ is the wave number and δ_y is the Dirac delta distribution at $y \in \mathbb{R}^3_+$, i.e., the propagating wave is generated by a point source located in the upper half-space.

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Moreover, the total field $u = u(\cdot; y, k)$ is assumed to satisfy the impedance boundary value condition

$$\frac{\partial u}{\partial x_3}(x) + \lambda_k u(x) = 0 \tag{2}$$

on $\mathbb{R}^3_0 := \mathbb{R}^2 \times \{0\}$, where $\lambda_k = \lambda_k(x)$ is an unknown realization of a real-valued random function with a bounded support. We assume that the wave number k is positive and λ_k is real-valued. Notice that in our model λ_k depends on k.

The classical problem with impedance boundary value condition in the half-space geometry is to understand what kind of surface waves appears on $\mathbb{R}^3_0$. Related to this, the uniqueness of the solution in many cases requires a special radiation condition [15,26]. In our case it can be shown that the classical Sommerfeld radiation condition

$$\left(\frac{\partial}{\partial r} - ik\right) u(x) = o(|x|^{-1}), \quad \text{as } |x| \to \infty \tag{3}$$

and uniformly in the sphere $x/|x| \in \mathbb{S}^2$, guarantees the uniqueness for a real-valued and compactly supported λ_k.

In the context of acoustics and sound propagation, the parameter λ_k is typically factorized as $\lambda_k = ik\beta$, where β is the acoustic admittance of the surface. It describes the ratio between the normal fluid velocity and the pressure at the surface. In this work we require $\mathrm{Re}\,(\beta) = 0$ in order to fulfill assumptions on λ_k. In acoustics, the boundary is said to be passive or non-absorbing.

Let U be an open and bounded set in $\mathbb{R}^3_+$. Our interest lies in the following inverse problem: *given the back-scattered field $u(y; y, k)$ for all $y \in U$ and $k > 0$, what information can be recovered regarding λ_k?* In other words, we take measurements generated by a single realization of λ_k and ask what properties the underlying random process had. The role of randomness in this treatise is to model the complex or chaotic micro-structure of λ_k. We do not focus on recovering λ_k exactly, but instead work towards determining some statistical properties regarding its probability distribution. We return to a more detailed formulation of this problem below.

Our work draws inspiration from [23], where inverse scattering was studied for a two-dimensional random Schrödinger equation $(\Delta + q + k^2)u = 0$. The potential q was assumed to be a Gaussian random function such that the covariance operator is a classical pseudodifferential operator [18]. The result in [23] shows that the backscattered field, obtained from a single realization of q, determines uniquely the principal symbol of the covariance operator of q. The statistical model for q assumes that the potential is locally isotropic and that the smoothness remains unchanged in spatial changes. However, the local variance is allowed to vary. A random field with such properties was called *microlocally isotropic*. This large class of random fields includes stochastic processes like the Brownian bridge or the Levy Brownian motion in the plane.

In the present treatise we generalize this concept to a class of random fields that are called *microlocally anisotropic*. Similar to [23] the covariance operator is assumed to be a pseudodifferential operator. However, the principal symbol is allowed to be direction-dependent. Hence, the correlation of the field is anisotropic while the smoothness remains unchanged in spatial changes.

Our main results in Theorems 2.5 and 2.7 relate to the unique recovery of the principal symbol [18] of the covariance of λ. In the isotropic case, the principal symbol can be fully recovered. If