ARTICLE IN PRESS
YJDEQ:8624

Available online at www.sciencedirect.com
ScienceDirect
J. Differential Equations ••• (••••) •••–•••
Journal of Differential Equations
www.elsevier.com/locate/jde

# On the fractional Fisher information with applications to a hyperbolic–parabolic system of chemotaxis

Rafael Granero-Belinchón

*Univ Lyon, Université Claude Bernard Lyon 1, CNRS UMR 5208, Institut Camille Jordan, 43 blvd. du 11 novembre 1918, F-69622 Villeurbanne cedex, France*

Received 2 September 2016; revised 6 November 2016

**Abstract**

We introduce new lower bounds for the fractional Fisher information. Equipped with these bounds we study a hyperbolic–parabolic model of chemotaxis and prove the global existence of solutions in certain dissipation regimes.


*MSC:* 35A01; 35M31; 35B65; 35Q92

*Keywords:* Fisher information; Weak solutions; Hyperbolic–parabolic system

## 1. Introduction

In this note we study the following system of partial differential equations

$$\begin{cases} \partial_t u = -\mu \Lambda^\alpha u + \partial_x (uq), \\ \partial_t q = \partial_x f(u), \end{cases} \quad \text{for } x \in \mathbb{T},\ t \geq 0, \tag{1}$$

where $\mathbb{T}$ denotes the 1-dimensional torus, $f$ is a smooth function, $\Lambda^\alpha = (-\Delta)^{\alpha/2}$ denotes the fractional Laplacian with $0 < \alpha \leq 2$ (see Appendix A for the expression as a singular integral and some properties) and $\mu \geq 0$ is a fixed constant.

*E-mail address:* granero@math.univ-lyon1.fr.

http://dx.doi.org/10.1016/j.jde.2016.11.028

This system was proposed by Othmers & Stevens [57] (see also Levine, Sleeman, Brian & Nilsen-Hamilton [39]) based on biological considerations as a model of the formation of new blood vessels from pre-existing blood vessels (in a process that is called tumor angiogenesis). In particular, in the previous system, $u$ is the density of vascular endothelial cells and $q = \partial_x \log(v)$ where $v$ is the concentration of the signal protein known as vascular endothelial growth factor (VEGF). As $f$ comes from the chemical kinetics of the system, it is commonly referred to as the *kinetic function*. The interested reader can refer to Bellomo, Li, & Maini [4] for a detailed exposition on tumor modelling. In the case where $f(u) = u^2/2$, equation (1) also appears as a viscous regularization of the dispersionless Majda–Biello model of the interaction of barotropic and equatorial baroclinic Rossby waves [50]. Another related model is the magnetohydrodynamic-Burgers system proposed by Fleischer & Diamond [27] (see also Jin, Wang & Xiong [36] and the references therein).

We address the existence of solutions and their qualitative properties in the case $0 < \alpha < 2$. In particular, among other results, we prove the global existence of weak solutions for $f(u) = u^r/r$, $1 \leq r \leq 2$ and $\alpha > 2 - r$. This topic is mathematically challenging due to the hyperbolic character of the equation for $q$. Indeed, at least formally, the velocity $q$ is one derivative less regular than $u$. So, the term $\partial_x(uq)$ is two derivatives less regular than $u$. This suggests that the diffusion given by the Laplacian ($\alpha = 2$) is somehow critical.

The main tool to achieve the results is a set of new inequalities for the generalized Fisher information (see [61] for a similar functional)

$$\mathcal{I}_\alpha = \int_{\mathbb{T}} (-\Delta)^{\alpha/2} u \Gamma(u) dx, \tag{2}$$

where $\Gamma$ is a smooth increasing function. This functional is a generalization of the classical Fisher information (also known as Linnik functional)

$$\mathcal{I}_2 = \int_{\mathbb{T}} -\Delta u \log(u) dx, \tag{3}$$

introduced in Fisher [26] (see also Linnik [48], McKean [51], Toscani [59,60], Villani [62]). The Fisher information appears commonly as the rate at which the Shannon's entropy[1] [56] (or, equivalently, the Boltzmann's H function)

$$\mathcal{S} = \int_{\mathbb{T}} u \log(u) dx, \tag{4}$$

is dissipated by diffusive semigroups as, for instance, the semigroup generated by the linear heat equation.

[1] To be completely precise, the original Shannon's entropy is $-\mathcal{S}$ and not $\mathcal{S}$ itself.

Please cite this article in press as: R. Granero-Belinchón, On the fractional Fisher information with applications to a hyperbolic–parabolic system of chemotaxis, J. Differential Equations (2016), http://dx.doi.org/10.1016/j.jde.2016.11.028