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Stochastic functional differential equations with infinite delay: Existence and uniqueness of solutions, solution maps, Markov properties, and ergodicity ☆

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Abstract

This work is devoted to stochastic functional differential equations (SFDEs) with infinite delay. First, existence and uniqueness of the solutions of such equations are examined. Because the solutions of the delay equations are not Markov, a viable alternative for studying further asymptotic properties is to use solution maps or segment processes. By examining solution maps, this work investigates the Markov properties as well as the strong Markov properties. Also obtained are adaptivity and continuity, mean-square boundedness, and convergence of solution maps from different initial data. This paper then examines the ergodicity of underlying processes and establishes existence of the invariant measure for SFDEs with infinite delay under suitable conditions.

MSC: 34K50; 60H10; 60J25; 60J60

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1. Introduction

Because time delays are ubiquitous, pervasive, and entrenched in everyday life, they have received considerable attention. In the context of dynamical systems, a class of such systems, namely, functional differential equations, has become an important focal point of research and investigation. The motivation stems from non-instant transmission phenomena, for example, high velocity fields in wind funnel experiments, or other memory processes, or biological applications (see [10,20,21,28]) such as species' growth or incubating time on disease models among many others. Theory of functional differential equations with infinite delay and its applications were developed in the 1970s and 1980s; see [6,11–14,18,29,30] and references therein. A comprehensive study of functional differential equations with infinite delay can be found in [14]. Recently, theories of functional differential equations with infinite delay including stability and their applications have attracted much of researchers' attention; see e.g., [1,4,6,8–10,21,24,34]. Because uncertainties are commonly encountered in many real systems and are often sources of instability [19], much work has been devoted to stochastic functional differential equations (SFDEs) and their applications; see, for example, [2,3,19,22–26,32]. Along this line, theory of SFDEs with infinite delay has also received much attentions; see [15,31,33,35,36] and references therein.

It is well known that the solutions of stochastic functional or delay differential equations are non-Markov since they depend on their history, so none of the properties of solutions based on the Markov property are applicable. To overcome this difficulty, Mohammed [26] examined solution maps of SFDEs with finite delay on appropriate phase spaces and proved that the solution maps have Markov property. Based on the Markov property of solution maps of SFDEs with finite delay, Bao et al. [2,3] examined the ergodicity. Since distributions of the solutions are the marginal distributions of the solution maps, the existence of stationary distributions of the solution maps implies that of the solutions. In fact, the solution maps possess many nice properties such as continuous semigroup of transformations; see [6,11–14,18,29,30]. However, as mentioned in [17], of fundamental importance for all approaches is the right choice of the phase space that in most cases is a Banach space of functions or equivalence classes of functions. For functional equations with finite delays, this is generally not a difficult problem. But for infinite delay equations the choice of an appropriate phase space is non-trivial; see [27].

Let us consider a SFDE with infinite delay of the form

$$dx(t) = f(x_t)dt + g(x_t)dw(t) \tag{1.1}$$

on $t \geq 0$ with the initial data $x_0 = \xi \in C_r$, where $x_t = x_t(\theta) =: \{x(t+\theta), -\infty < \theta \leq 0\}$, $f : C_r \to \mathbb{R}^n$ and $g : C_r \to \mathbb{R}^{n\times m}$ are Borel measurable, $w(t)$ is an m-dimensional Brownian motion. To show the dependence of the solution $x(t)$ on the initial data, we also write $x(t)$ as $x(t;\xi)$ or $x(t;t_0,\xi)$ if the initial segment is $x_{t_0} = \xi$ at t_0. Correspondingly, we also write x_t as $x_t(\xi)$ or $x_t(t_0, x_{t_0})$. When $-\infty < \theta \leq 0$ is considered, $x_t(\theta)$ can be written as $x_t(\theta;\xi)$ or $x_t(\theta;t_0,\xi)$ if the initial segment is ξ at t_0. If (1.1) has a solution $x(t;t_0,\xi)$ with the initial segment ξ at t_0, then $x_t(t_0;\xi)$ is called the solution map. In this paper, aiming at examining such asymptotic properties as mean-square boundedness, convergence of different solutions from different initial data, and

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