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Nonlocal diffusion second order partial differential equations

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Abstract

The paper deals with a second order integro-partial differential equation in $\mathbb{R}^n$ with a nonlocal, degenerate diffusion term. Nonlocal conditions, such as the Cauchy multipoint and the weighted mean value problem, are investigated. The existence of periodic solutions is also studied. The dynamic is transformed into an abstract setting and the results come from an approximation solvability method. It combines a Schauder degree argument with an Hartman-type inequality and it involves a Scorza-Dragoni type result. The compact embedding of a suitable Sobolev space in the corresponding Lebesgue space is the unique amount of compactness which is needed in this discussion. The solutions are located in bounded sets and they are limits of functions with values in finitely dimensional spaces.

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1. Introduction

This paper deals with the partial integro-differential equation

$$u_{tt} = cu_t + bu(t,\xi) + u(t,\xi)\int_\Omega k(\xi,\eta)u(t,\eta)d\eta + h(t,u(t,\xi)) \tag{1.1}$$

where $\Omega \subset \mathbb{R}^n$ ($n \geq 2$) is an open, bounded domain with C^1-boundary, b and c are given constants, $h\colon [0,T]\times\mathbb{R} \to \mathbb{R}$ and $k : \Omega\times\Omega \to \mathbb{R}$ are given functions.

If the integral term is replaced by the usual Laplace operator Δu, then according to the values of c and b, (1.1) becomes the damped wave equation or the telegraph equation or the Klein–Gordon equation and it is a model for many phenomena. For example, it governs the propagation of electro-magnetic waves in an electrically conducting medium, the motion of a string or a membrane with external damping, the evolution of visco-elastic fluids influenced by Maxwell theory and the heat propagation in a thermally conducting medium (see e.g. [15,18] and the references therein).

The classical diffusion equation implies an infinite velocity propagation of information; namely, a change in temperature or concentration in some point of the domain is instantaneously felt everywhere. In recent years, to circumvent this drawback, many authors have proposed alternatives to describe heat and mass transfer. A continuous diffusion coefficient which vanishes when $u = 0$ is assumed in [12], to obtain a degenerate process with finite speed of propagation. Alternatively, a fractional diffusion term is considered in [9], instead of the standard Laplace operator.

Diffusion operators such as the integral contained in (1.1) introduce a memory effect in the equation and are able to capture long distance interactions into the process that occur in a number of applications; hence they are frequently preferable than the classical punctual diffusions such as Laplace operator. As a consequence, several investigations recently appeared, for first order dynamics, which include an integral diffusion term (see e.g. [14,16] and references therein). As far as we know, this is the first paper investigating a nonlocal diffusion second order equation.

The presence of u as a non-constant diffusion coefficient (similarly as in [20]) means that the diffusion has a degenerate nature as in [12]. This is an expected behavior in several contexts. This term allows a super-linear growth of the right hand side of the equation, also often appearing in many applications.

The main aim of this paper is to start a theory on some important classes of solutions of (1.1). We assume the following conditions

(1) h is continuous and the partial derivative $\frac{\partial h}{\partial z}\colon [0,T]\times\mathbb{R}\to\mathbb{R}$ is continuous and there is a positive constant N, such that

$$\left|\frac{\partial h(t,z)}{\partial z}\right| \leq N \text{ for all } (t,z)\in[0,T]\times\mathbb{R}.$$

(2) $k \in W^{2,1}(\Omega\times\Omega,\mathbb{R})$ and $\max\{\sup_{(\xi,\eta)\in\Omega\times\Omega}|k(\xi,\eta)|, \sup_{(\xi,\eta)\in\Omega\times\Omega}\|Dk(\xi,\eta)\|_{\mathbb{R}^n}\} = K < \infty$, where the symbol D stands for the derivative (i.e. the gradient) with respect to the variables in the vector $\xi\in\Omega$.

(3) $b \geq N + \sqrt{6\delta K|\Omega|}$ where $|\Omega|$ denotes the Lebesgue measure of Ω and $\delta = \max_{t\in[0,T]}|h(t,0)|$.