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On the Dirichlet problem for hypoelliptic evolution equations: Perron–Wiener solution and a cone-type criterion

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Abstract

We show how to apply harmonic spaces potential theory in the study of the Dirichlet problem for a general class of evolution hypoelliptic partial differential equations of second order. We construct Perron–Wiener solution and we provide a sufficient condition for the regularity of the boundary points. Our criterion extends and generalizes the classical parabolic-cone criterion for the Heat equation due to Effros and Kazdan.

MSC: 35H10; 35K70; 35K65; 31D05; 35D99; 35J25

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1. Introduction

The aim of this paper is to prove the existence of a generalized solution in the sense of Perron–Wiener to the Dirichlet problem and to provide a sufficient condition for the regularity of the boundary points for a wide class of evolution equations.

More precisely, we consider second order partial differential operators of the following type

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$$\mathcal{L} = \sum_{i,j=1}^{N} a_{ij}(z)\partial_{x_i x_j} + \sum_{i=1}^{N} b_i(z)\partial_{x_i} - \partial_t, \tag{1}$$

in a strip

$$S = \{z = (x,t) \in \mathbb{R}^{N+1} \mid x \in \mathbb{R}^N,\ T_1 < t < T_2\},$$

with $-\infty \le T_1 < T_2 \le +\infty$.

The coefficients $a_{ij} = a_{ji}$ and b_i are smooth and the characteristic form of the operator is nonnegative definite and non-totally degenerate, i.e.,

$$\sum_{i,j=1}^{N} a_{ij}(z)\xi_i\xi_j \ge 0, \qquad \forall z \in S,\ \forall \xi = (\xi_1, \dots, \xi_N) \in \mathbb{R}^N,$$

and

$$\sum_{i=1}^{N} a_{ii}(z) > 0 \qquad \forall z \in S.$$

Finally, we assume the *hypoellipticity* of $\mathcal{L} - \beta$ and of $\mathcal{L}^*$, for every constant $\beta \ge 0$, and the existence of a well-behaved *fundamental solution* Γ for $\mathcal{L}$,

$$(z,\zeta) \longmapsto \Gamma(z,\zeta),$$

satisfying the following properties:

(i) $\Gamma(\cdot,\zeta)$ belongs to $L^1_{\text{loc}}(S)$ and $\mathcal{L}(\Gamma(\cdot,\zeta)) = -\delta_\zeta$, where δ_ζ denotes the Dirac measure at $\{\zeta\}$, for every $\zeta \in S$.

(ii) For every $\varphi \in C_0^\infty(\mathbb{R}^N)$ and for every $(x_0,\tau) \in S$,

$$\int_{\mathbb{R}^N} \Gamma(x,t,\xi,\tau)\varphi(\xi)\,d\xi \to \varphi(x_0), \qquad \text{as } (x,t) \to (x_0,\tau),\ t > \tau.$$

(iii) $\Gamma \in C^\infty\Big(\{(z,\zeta) \in \mathbb{R}^{N+1} \times \mathbb{R}^{N+1} \mid z \ne \zeta\}\Big)$.

(iv) $\Gamma \ge 0$ and $\Gamma(x,t,\xi,\tau) > 0$ if and only if $t > \tau$. Moreover, for every fixed $z \in S$, $\limsup_{\zeta \to z} \Gamma(z,\zeta) = \infty$.

(v) $\Gamma(z,\zeta) \to 0$ for $\zeta \to \infty$ uniformly for $z \in K$, compact set of S, and, analogously, $\Gamma(z,\zeta) \to 0$ for $z \to \infty$ uniformly for $\zeta \in K$, compact set of S.

(vi) $\exists\, C > 0$ such that for any $z = (x,t) \in S$ we have

$$\int_{\mathbb{R}^N} \Gamma(z;\xi,\tau)\,d\xi \le C \quad \text{if} \quad t > \tau.$$

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