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Averaged controllability of parameter dependent conservative semigroups

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Abstract

We consider the problem of averaged controllability for parameter depending (either in a discrete or continuous fashion) control systems, the aim being to find a control, independent of the unknown parameters, so that the average of the states is controlled. We do it in the context of conservative models, both in an abstract setting and also analysing the specific examples of the wave and Schrödinger equations.

Our first result is of perturbative nature. Assuming the averaging probability measure to be a small parameter-dependent perturbation (in a sense that we make precise) of an atomic measure given by a Dirac mass corresponding to a specific realisation of the system, we show that the averaged controllability property is achieved whenever the system corresponding to the support of the Dirac is controllable.

Similar tools can be employed to obtain averaged versions of the so-called *Ingham inequalities*.

Particular attention is devoted to the 1d wave equation in which the time-periodicity of solutions can be exploited to obtain more precise results, provided the parameters involved satisfy Diophantine conditions ensuring the lack of resonances.

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1. Introduction and main results

1.1. Problem formulation

This paper is devoted to analysing the following question: *Given a system depending on a random variable, is it possible to find a control such that the average or expected value of the output of the system is controlled?*

The problem is addressed in the context of the abstract system

$$\dot{y}_\zeta = A_\zeta y_\zeta + B_\zeta u \,, \tag{1.1a}$$

with the parameter dependent initial condition

$$y_\zeta(0) = \mathrm{y}^i_\zeta \,, \tag{1.1b}$$

where $\zeta \in \mathbb{R}$ is a random variable following a probability law η, A_ζ is an operator on X, the state space, B_ζ is a control operator, $y_\zeta(t) \in X$ is the parameter dependent state variable, and $u(t) \in U$ is the control variable (independent of the parameter ζ), U being the control space.

Given $T > 0$, the problem of *exact averaged controllability* consists in analysing whether, for every family of parameter dependent initial conditions $\mathrm{y}^i_\zeta \in X$ and every final target $\mathrm{y}^f \in X$, there exists a control $u \in L^2([0,T], U)$ (independent of the parameter ζ) such that:

$$\int_{\mathbb{R}} y_\zeta(T)\, \mathrm{d}\eta_\zeta = \mathrm{y}^f \,. \tag{1.2}$$

One can also address the weaker *approximate averaged control problem*, in which, for every $\varepsilon > 0$, one aims to find a control $u \in L^2([0,T], U)$ such that:

$$\left\| \int_{\mathbb{R}} y_\zeta(T)\, \mathrm{d}\eta_\zeta - \mathrm{y}^f \right\|_X^2 \leqslant \varepsilon \,. \tag{1.3}$$

In both (1.2) and (1.3), y_ζ is the solution of (1.1) with initial Cauchy condition y^i_ζ and control u.

As we shall see, the averaged controllability properties will significantly depend on the nature of the averaging measure η.

It is easy to see that averaged control problems cannot be handled by classical methods. Indeed,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_{\mathbb{R}} y_\zeta(t)\, \mathrm{d}\eta_\zeta \right) = \int_{\mathbb{R}} A_\zeta y_\zeta \, \mathrm{d}\eta_\zeta + \left(\int_{\mathbb{R}} B_\zeta \, \mathrm{d}\eta_\zeta \right) u \,,$$