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Asymptotic stability of singularly perturbed differential equations

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Abstract

Asymptotic stability is examined for singularly perturbed ordinary differential equations that may not possess a natural split into fast and slow motions. Rather, the right hand side of the equation is comprised of a singularly perturbed component and a regular one. The limit dynamics consists then of Young measures, with values being invariant measures of the fast contribution, drifted by the slow one. Relations between the asymptotic stability of the perturbed system and the limit dynamics are examined, and a Lyapunov functions criterion, based on averaging, is established.

Keywords: Asymptotic stability; Singular perturbations; Invariant measures; Young measures

1. Introduction

The paper examines the Lyapunov asymptotic stability of a motion $x = x(t)$, when the dynamics is governed by a singularly perturbed differential equation of the form

$$\frac{dx}{dt} = \frac{1}{\varepsilon}F(x) + G(x), \tag{1.1}$$

with $\varepsilon > 0$ a small real parameter, and $x \in R^n$.

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Notice that the dynamics in (1.1) does not exhibit a split into a fast and a slow dynamics. Indeed, we examine a general case where such a split is either not tractable or may not exist. The Tikhonov form, which is the classical singularly perturbed model, is a particular case.

We can, however, identify in system (1.1) a slow and a fast contribution to the dynamics, namely, the components $G(x)$ and, respectively, $\frac{1}{\varepsilon}F(x)$ (compare with Tao, Owhadi and Marsden [15], and with Artstein, Kevrekidis, Slemrod and Titi [3]). Accordingly, the differential equation

$$\frac{dx}{ds} = F(x) \tag{1.2}$$

is referred to as the *fast part* of (1.1), and $G(x)$ is referred to as the *drift*. Notice that when moving from (1.1) to (1.2), we have changed the time scale, with $t = \varepsilon s$. It was proved in [3] (we recall later the details), that under quite general conditions, the dynamics of (1.1) converge, as $\varepsilon \to 0$, to Young measures, namely, probability measure-valued maps, whose values are invariant measures of (1.2), drifted by the slow component of the equation.

The classical model of singular perturbation, the Tikhonov model, takes the form

$$\begin{aligned} \frac{dy}{dt} &= f(y,z) \\ \frac{dz}{dt} &= \frac{1}{\varepsilon}g(y,z), \end{aligned} \tag{1.3}$$

with $y \in R^m$ and $z \in R^k$, see O'Malley [12,13], Tikhonov, Vasiléva and Sveshnikov [16], Verhulst [18], and Wasow [19]. It is clear that (1.3) is a particular case of (1.1) where $x = (y, z)$. In (1.3) one can identify coordinates (the z-coordinates) that move fast, while the y-coordinates move regularly. In particular, the fast equation is the z-equation when y is held fixed. The system (1.3), including stability results for it, is well studied and we allude to these results later in the paper.

Throughout the paper we adopt the following.

Assumption 1.1. *The function $F(\cdot)$ and $G(\cdot)$ are continuous. The solutions, say $x(\cdot)$, of the fast equation (1.2), are determined uniquely by the initial data, say $x(s_0) = x_0$, and stay bounded for $s \ge s_0$, uniformly for x_0 in a bounded set.*

Our main contributions are to identify the appropriate notions of asymptotic stability of the perturbed system and its limit dynamics; to examine the relationships among them; and to offer a Lyapunov functions criterion for the asymptotic stability, a criterion that relies on averaging with respect to the invariant measures.

In the next section we recall some facts about invariant measures and about Young measures, and restate the result concerning the form of the limit dynamics of (1.1). In Section 3 we display the definitions of asymptotic stability of the perturbed system and its limit dynamics. The asymptotic stability does not require that (1.1) be asymptotically stable when ε is fixed. Relations among the various definitions of asymptotic stability, along with some examples demonstrating the possible stability properties, are given in Section 4. In section 5 we offer a Lyapunov functions sufficient condition for the asymptotic stability of the limiting measure-valued dynamics.

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