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Journal of Differential Equations

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On bifurcation delay: An alternative approach using Geometric Singular Perturbation Theory

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Received 14 April 2016; revised 15 September 2016

Abstract

To explain the phenomenon of bifurcation delay, which occurs in planar systems of the form $\dot{x} = \epsilon f(x, z, \epsilon)$, $\dot{z} = g(x, z, \epsilon)z$, where $f(x, 0, 0) > 0$ and $g(x, 0, 0)$ changes sign at least once on the x-axis, we use the Exchange Lemma in Geometric Singular Perturbation Theory to track the limiting behavior of the solutions. Using the trick of extending dimension to overcome the degeneracy at the turning point, we show that the limiting attracting and repulsion points are given by the well-known entry-exit function, and the minimum of z on the trajectory is of order $\exp(-1/\epsilon)$. Also we prove smoothness of the return map up to arbitrary finite order in ϵ.

MSC: 34E20; 37C10

Keywords: Bifurcation delay; Pontryagin delay; Delay of instability; Entry-exit function; Slow–fast system; Exchange Lemma

1. Introduction

Consider the planar system

$$\begin{aligned} \dot{x} &= \epsilon f(x, z, \epsilon) \\ \dot{z} &= g(x, z, \epsilon)z \end{aligned} \tag{1.1ϵ}$$

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[1] This work was mainly done at the Ohio State University when the author served as a lecturer.

http://dx.doi.org/10.1016/j.jde.2016.10.022
0022-0396/

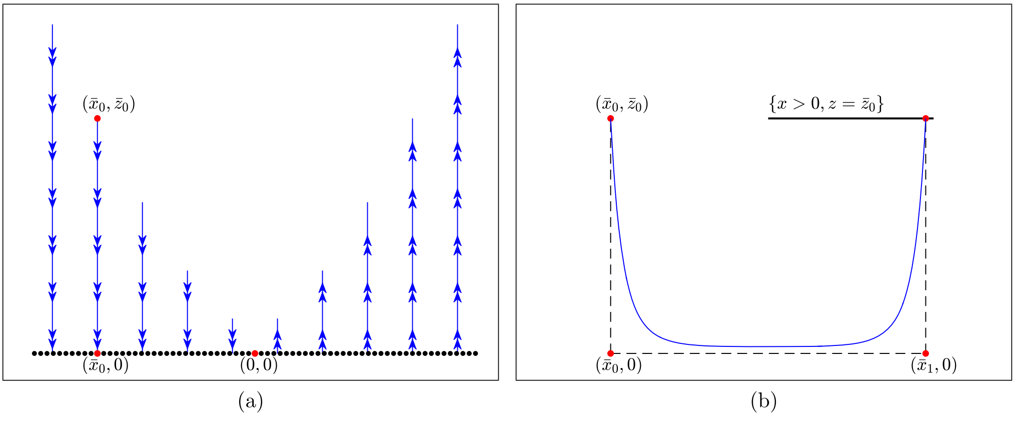

Fig. 1. (a) When $\epsilon = 0$, the x-axis is a line of equilibria for (1.1ϵ). The trajectory starting at (x_0, z_0) goes straight down to the x-axis. At the turning point $(x, z) = (0, 0)$, the x-axis changes from attracting to repelling. (b) When $\epsilon > 0$ and small, the trajectory of (1.1ϵ) starting at (x_0, z_0) first tends to the x-axis, and then it turns at $(x_0, 0)$. The trajectory turns again when it is near the point $(x_1, 0)$ satisfying $\int_{x_0}^{x_1} \frac{g(x,0,0)}{f(x,0,0)}\, dx = 0$.

with $x \in \mathbb{R}$, $z \in \mathbb{R}$, f and g are C^1 functions satisfying

$$f(x, 0, 0) > 0; \quad g(x, 0, 0) < 0 \text{ for } x < 0 \quad \text{and} \quad g(x, 0, 0) > 0 \text{ for } x > 0. \tag{1.2}$$

Note that (1.1ϵ) is a *slow–fast* system [10,12] with fast variable z and slow variable x. Fix any $x_0 < 0$ and choose $z_0 > 0$ small enough so that $g(x_0, z, 0) < 0$ for all $z \in [0, z_0]$. When $\epsilon = 0$, it is clear that the trajectory starting at (x_0, z_0) goes straight to $(x_0, 0)$. The x-axis is attracting when $x < 0$ and repelling when $x > 0$ since $g(x, 0, 0)$ changes sign at $x = 0$. For $\epsilon > 0$, besides being attracted by the x-axis, the trajectory also moves right at speed of order ϵ. After the trajectory passes $x = 0$, the x-axis becomes repelling, so the trajectory tends to move away from the x-axis. See Fig. 1a. However, it is well known that, for small $\epsilon > 0$, the trajectory does not immediately leave the vicinity of the x-axis after crossing the origin. Instead, the trajectory stays at the x-axis until it is near the point $(x_1, 0)$ that satisfies

$$\int_{x_0}^{x_1} \frac{g(x, 0, 0)}{f(x, 0, 0)}\, dx = 0. \tag{1.3}$$

See Fig. 1b. This phenomenon has been called "bifurcation delay" [2], "Pontryagin delay" [15], or "delay of instability" [14], and the function $x_0 \mapsto x_1$ implicitly defined by (1.3) is called the entry-exit [1] or way in-way out [5] function.

Bifurcation delay has been studied by various methods in the literature, including asymptotic expansion [9,15], comparison to solutions constructed by separation of variables [16], gradient estimates using the variational equation [6], and the blow-up method of geometric singular perturbation theory [7].

Our results stated below are included in the literature, but the proof in this note provides a new approach using geometric singular perturbation theory.

Please cite this article in press as: T.-H. Hsu, On bifurcation delay: An alternative approach using Geometric Singular Perturbation Theory, J. Differential Equations (2016), http://dx.doi.org/10.1016/j.jde.2016.10.022