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# Stable determination of sound-hard polyhedral scatterers by a minimal number of scattering measurements

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**Abstract**

The aim of the paper is to establish optimal stability estimates for the determination of sound-hard polyhedral scatterers in $\mathbb{R}^N$, $N \geq 2$, by a minimal number of far-field measurements. This work is a significant and highly nontrivial extension of the stability estimates for the determination of sound-soft polyhedral scatterers by far-field measurements, proved by one of the authors, to the much more challenging sound-hard case.

The admissible polyhedral scatterers satisfy minimal a priori assumptions of Lipschitz type and may include at the same time solid obstacles and screen-type components. In this case we obtain a stability estimate with $N$ far-field measurements. Important features of such an estimate are that we have an explicit dependence on the parameter $h$ representing the minimal size of the cells forming the boundaries of the admissible polyhedral scatterers, and that the modulus of continuity, provided the error is small enough with respect to $h$, does not depend on $h$. If we restrict to $N = 2, 3$ and to polyhedral obstacles, that is to polyhedra, then we obtain stability estimates with fewer measurements, namely first with $N - 1$ measurements and then with a single measurement. In this case the dependence on $h$ is not explicit anymore and the modulus of continuity depends on $h$ as well.



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## 1. Introduction

A set $\Sigma \subset \mathbb{R}^N$, $N \geq 2$, is called a *scatterer* if it is a compact set such that $\mathbb{R}^N \setminus \Sigma$ is connected. A scatterer is said to be an obstacle if it is the closure of an open set and it is said to be a screen if its interior is empty.

If an incident time-harmonic acoustic wave encounters a scatterer then it is perturbed through the creation of a scattered or reflected wave. The total wave is given by the superposition of the incident and the scattered wave and it is characterized by the total field $u$, solution to the following exterior boundary value problem

$$\begin{cases} \Delta u + k^2 u = 0 & \text{in } \mathbb{R}^N \setminus \Sigma \\ u = u^i + u^s & \text{in } \mathbb{R}^N \setminus \Sigma \\ B.C. & \text{on } \partial\Sigma \\ \lim_{r\to\infty} r^{(N-1)/2}\left(\dfrac{\partial u^s}{\partial r} - \mathrm{i}ku^s\right) = 0 & r = \|x\|. \end{cases}$$

Here $k > 0$ in the reduced wave equation, or Helmholtz equation, is the wavenumber and $u^i$ is the incident field, that is the field of the incident wave. The incident field is usually an entire solution of the Helmholtz equation, here we shall always assume that the incident wave is a time-harmonic plane wave with direction of propagation $v \in \mathbb{S}^{N-1}$, that is $u^i(x) = \mathrm{e}^{\mathrm{i}kx\cdot v}$, $x \in \mathbb{R}^N$. Instead $u^s$ is the scattered field, that is the field of the scattered wave. The last limit is the Sommerfeld radiation condition and corresponds to the fact that the scattered wave is radiating. Moreover it implies that the scattered field has the following asymptotic behavior

$$u^s(x) = \frac{\mathrm{e}^{\mathrm{i}k\|x\|}}{\|x\|^{(N-1)/2}}\left\{u_\infty(\hat{x}) + O\left(\frac{1}{\|x\|}\right)\right\},$$

where $\hat{x} = x/\|x\| \in \mathbb{S}^{N-1}$ and $u_\infty$ is the so-called *far-field pattern* of $u^s$. We shall also write $u_\infty(\hat{x}; \Sigma, k, v)$ to specify its dependence on the observation direction $\hat{x} \in \mathbb{S}^{N-1}$, the scatterer $\Sigma$, the wavenumber $k > 0$ and the direction of propagation of the incident field $v \in \mathbb{S}^{N-1}$.

Finally, the boundary condition on the boundary of $\Sigma$ depends on the physical properties of the scatterer $\Sigma$. If $\Sigma$ is *sound-soft*, then $u$ satisfies a homogeneous Dirichlet condition whereas if $\Sigma$ is *sound-hard* we have a homogeneous Neumann condition. We remark that other conditions such as the impedance boundary condition or transmission conditions for penetrable scatterers may be of interest for the applications.

The inverse scattering problem consists of recovering the scatterer $\Sigma$ by its corresponding far-field measurements for one or more incident waves. Such an inverse problem is of fundamental importance to many areas of science and technology including radar and sonar applications, geophysical exploration, medical imaging and nondestructive testing. For a general introduction on this inverse problem see for instance [4,12].

Physically, a far-field measurement is obtained by sending an incident plane wave and measuring the scattered wave field faraway at every possible observation directions, namely by measuring the far-field pattern $u_\infty$ of $u^s$.

If we measure the far-field pattern for just one incident plane wave, then we say that we use a single far-field measurement. We can obtain multiple far-field measurements by sending