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Spectral decomposition of fractional operators and a reflected stable semigroup ☆

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Abstract

In this paper, we provide the spectral decomposition in Hilbert space of the $\mathcal{C}_0$-semigroup P and its adjoint $\widehat{P}$ having as generator, respectively, the Caputo and the right-sided Riemann–Liouville fractional derivatives of index $1 < \alpha < 2$. These linear operators, which are non-local and non-self-adjoint, appear in many recent studies in applied mathematics and also arise as the infinitesimal generators of some substantial processes such as the reflected spectrally negative α-stable process. Our approach relies on intertwining relations that we establish between these semigroups and the semigroup of a Bessel type process whose generator is a self-adjoint second order differential operator. In particular, from this commutation relation, we characterize the positive real axis as the continuous point spectrum of P and provide a power series representation of the corresponding eigenfunctions. We also identify the positive real axis as the residual spectrum of the adjoint operator $\widehat{P}$ and elucidate its role in the spectral decomposition of these operators. By resorting to the concept of continuous frames, we proceed by investigating the domain of the spectral operators and derive two representations for the heat kernels of these semigroups. As a by-product, we also obtain regularity properties for these latter and also for the solution of the associated Cauchy problem.
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1. Introduction

Fractional calculus, in which derivatives and integrals of fractional order are defined and studied, is nearly as old as the classical calculus of integer orders. Ever since the first inquisition by L'Hopital and Leibniz in 1695, there has been an enormous amount of study on this topic for more than three centuries, with many mathematicians having suggested their own definitions that fit the concept of a non-integer order derivative. Among the most famous of these definitions are the Riemann–Liouville fractional derivative and the Caputo derivative, the latter being a reformulation of the former in order to use integer order initial conditions to solve fractional order differential equations. In this context, it is natural to consider the following Cauchy problem, for a smooth function f on $x > 0$,

$$\begin{cases} \frac{d}{dt}u(t,x) = \mathbf{D}_\alpha u(t,x) \\ u(0,x) = f(x), \end{cases} \tag{1.1}$$

where, for any $1 < \alpha < 2$, the linear operator $\mathbf{D}_\alpha$ is either the Caputo α-fractional derivative

$$\mathbf{D}_\alpha f(x) = {}^C D_+^\alpha f(x) = \int_0^x \frac{f^{([\alpha]+1)}(y)}{(x-y)^{\alpha-[\alpha]}} \frac{dy}{\Gamma([\alpha]+1-\alpha)}, \tag{1.2}$$

with, for any $k = 1, 2, \dots,$ $f^{(k)}(x) = \frac{d^k}{dx^k} f(x)$ stands for the k-th derivative of f, or, the right-sided Riemann–Liouville (RL) derivative

$$\mathbf{D}_\alpha f(x) = D_-^\alpha f(x) = \left(\frac{d}{dx}\right)^{[\alpha]+1} \int_x^\infty \frac{f(y)(y-x)^{[\alpha]-\alpha}}{\Gamma([\alpha]+1-\alpha)} dy, \tag{1.3}$$

with $[\alpha]$ representing the integral part of α. We point out that when $\alpha = 2$, in both cases, $\mathbf{D}_2 f(x) = \frac{1}{2} f^{(2)}(x)$ is a second order differential operator.

In this paper, we aim at providing the spectral representation in $\mathrm{L}^2(\mathbb{R}_+)$ Hilbert space and regularities properties of the solution to the Cauchy problem (1.1).

The motivations underlying this study are several folds. On the one hand, the last three decades have witnessed the most intriguing leaps in engineering and scientific applications of such fractional operators, including but not limited to population dynamics, chemical technology, biotechnology and control of dynamical systems, and, we refer to the monographs of Kilbas et al. [19], Meerschaert and Sikorskii [25] and Sankaranarayanan [39] for excellent and recent accounts on fractional operators. On the other hand, some recent interesting studies have revealed that the linear operator ${}^C D_+^\alpha$ is the infinitesimal generator of $P = (P_t)_{t \geq 0}$ the Feller semigroup corresponding to the so-called spectrally negative reflected α-stable process, see e.g. [2,5,35]. We will provide the formal definition of this process and semigroup in Section 2, and, we simply point out that the reflected Brownian motion is obtained in the limiting case $\alpha = 2$. The reflected

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