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Gradient bounds for a thin film epitaxy equation

Dong Li [a], Zhonghua Qiao [b,*], Tao Tang [c,d]

[a] Department of Mathematics, University of British Columbia, 1984 Mathematics Road, Vancouver, BC, V6T 1Z2, Canada

[b] Department of Applied Mathematics, The Hong Kong Polytechnic University, Hung Hom, Hong Kong

[c] Department of Mathematics, South University of Science and Technology, Shenzhen, Guangdong 518055, China

[d] Department of Mathematics, Hong Kong Baptist University, Kowloon, Hong Kong

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Abstract

We consider a gradient flow modeling the epitaxial growth of thin films with slope selection. The surface height profile satisfies a nonlinear diffusion equation with biharmonic dissipation. We establish optimal local and global wellposedness for initial data with critical regularity. To understand the mechanism of slope selection and the dependence on the dissipation coefficient, we exhibit several lower and upper bounds for the gradient of the solution in physical dimensions $d \le 3$.

Keywords: Epitaxy; Thin film; Maximum principle; Gradient bound

1. Introduction

Let $\nu > 0$. Consider

$$\partial_t h = \nabla \cdot ((|\nabla h|^2 - 1)\nabla h) - \nu \Delta^2 h \tag{1.1}$$

and the 1D version

* Corresponding author.
E-mail addresses: dli@math.ubc.ca (D. Li), zhonghua.qiao@polyu.edu.hk (Z. Qiao), tangt@sustc.edu.cn (T. Tang).

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$$h_t = (h_x^3 - h_x)_x - \nu h_{xxxx}. \tag{1.2}$$

Eq. (1.1) is a nonlinear diffusion equation which models the epitaxial growth of thin films. It is posed on the spatial domain Ω which can either be the whole space $\mathbb{R}^d$, the L-periodic torus ($L > 0$ is a parameter corresponding to the size of the system) $\mathbb{R}^d/L\mathbb{Z}^d$, or a finite domain in $\mathbb{R}^d$ with suitable boundary conditions. In this work for simplicity we shall be mainly concerned with the 2π-periodic case $\Omega = \mathbb{T}^d = \mathbb{R}^d/2\pi\mathbb{Z}^d$ but our results can be easily generalized to other settings. The function $h = h(t,x) : \mathbb{R} \times \Omega \to \mathbb{R}$ represents the scaled height of a thin film and $\nu > 0$ is positive parameter which is sometimes called the diffusion coefficient. Typically in numerical simulations one is interested in the regime where ν is small so that the nonlinear effects become dominant. The 1D version (1.2) is connected with the Cahn–Hilliard equation:

$$\partial_t u = \Delta(u^3 - u) - \nu\Delta^2 u$$

through the identification $u = \partial_x h$. This connection breaks down for dimension $d \geq 2$.

Define the energy

$$E(h) = \int_\Omega \Big(\frac{1}{4}(|\nabla h|^2 - 1)^2 + \frac{\nu}{2}|\Delta h|^2\Big)dx. \tag{1.3}$$

The equation (1.1) can be regarded as a gradient flow of the energy functional $E(h)$ in $L^2(\Omega)$. In fact, it is easy to check that

$$\frac{d}{dt}E(h) = -\|\partial_t h\|_2^2, \tag{1.4}$$

i.e. the energy is always decreasing in time as far as smooth solutions are concerned. Alternatively one can derive the energy law from (1.1) by multiplying both sides by $\partial_t h$ and integrating by parts. The first term in (1.3) models the Ehrlich–Schowoebel effect [3,12,13]. Formally speaking it forces the slope of the thin film $|\nabla h| \approx 1$. For this reason Eq. (1.1) is often called the growth equation with slope selection. On the other hand, in the literature there are also models "without slope selection", such as

$$\partial_t h = -\nabla \cdot \big(\frac{1}{1 + |\nabla h|^2}\nabla h\big) - \nu\Delta^2 h. \tag{1.5}$$

Heuristically speaking, if in (1.5) the slope $|\nabla h|$ is small, then

$$\frac{1}{1 + |\nabla h|^2} \approx 1 - |\nabla h|^2$$

and one recovers the nonlinearity in (1.1). However this line of argument seems only reasonable when $|\nabla h| \ll 1$ which is a typical transient regime and not so appealing physically. Indeed the long time interfacial dynamics governed by (1.1) and (1.5) can be quite different, see for example the discussion in [5]. The second term in (1.3) corresponds to the fourth-order diffusion in (1.1). It has a stabilizing effect both theoretically and numerically.