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The evolution to localized and front solutions in a non-Lipschitz reaction–diffusion Cauchy problem with trivial initial data

J.C. Meyer [1], D.J. Needham *

School of Mathematics, Watson Building, University of Birmingham, Birmingham, B15 2TT, UK

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Abstract

In this paper, we establish the existence of spatially inhomogeneous classical self-similar solutions to a non-Lipschitz semi-linear parabolic Cauchy problem with trivial initial data. Specifically we consider bounded solutions to an associated two-dimensional non-Lipschitz non-autonomous dynamical system, for which, we establish the existence of a two-parameter family of homoclinic connections on the origin, and a heteroclinic connection between two equilibrium points. Additionally, we obtain bounds and estimates on the rate of convergence of the homoclinic connections to the origin.

MSC: 35K58; 34C37

Keywords: Semi-linear parabolic PDE; Self-similar; Non-Lipschitz; Homoclinic connection; Heteroclinic connection

1. Introduction

In this paper, we study classical bounded solutions $u : \mathbb{R} \times [0, T] \to \mathbb{R}$ to the non-Lipschitz semi-linear parabolic Cauchy problem

* Corresponding author.
E-mail addresses: J.C.Meyer@bham.ac.uk (J.C. Meyer), D.J.Needham@bham.ac.uk (D.J. Needham).
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$$u_t - u_{xx} = u|u|^{p-1} \quad \text{on } \mathbb{R} \times (0, T], \tag{1}$$

$$u = 0 \quad \text{on } \mathbb{R} \times \{0\}, \tag{2}$$

with $0 < p < 1$ and $T > 0$ (which we henceforth refer to as [CP]). The primary achievement of the paper is the establishment of the existence of a two-parameter family of localized spatially inhomogeneous solutions to [CP] for which $u(x, t) \to 0$ as $|x| \to \infty$ uniformly for $t \in [0, T]$; the secondary achievement of the paper is the establishment of front solutions to [CP], which approach $\pm(1 - p)^{1/(1-p)} t^{1/(1-p)}$ as $|x| \to \pm\infty$ uniformly for $t \in [0, T]$. We note here that for $p \geq 1$ in (1), the unique bounded classical solution with initial data (2) is the trivial solution, see for example [1, Theorem 4.5].

Qualitative properties of non-negative (non-positive) solutions to (1) when $0 < p < 1$, with non-negative (non-positive) initial data, and for which $u(x, t)$ is bounded as $|x| \to \infty$ uniformly for $t \in [0, T]$, have been determined in [2–6]. However, we note that any non-negative (non-positive) classical bounded solution to [CP] must be spatially homogeneous for $t \in [0, T]$, see for example [2, Corollary 2.6]. Thus, the solutions constructed in this paper are two signed on $\mathbb{R} \times [0, T]$. The authors are currently unaware of any studies of two signed solutions to (1)–(2) with $0 < p < 1$. Generic local results for spatial homogeneity of solutions to semi-linear parabolic Cauchy problems with homogeneous initial data depend upon uniqueness results, see for example, [6]. For results concerning the related problem of asymptotic homogeneity (in general, asymptotic symmetry) as $t \to \infty$ of non-negative (non-positive) global solutions to semi-linear parabolic Cauchy problems, we refer the reader to the survey article [7].

Non-negative (non-positive), spatially inhomogeneous solutions to (1) for $p > 1$ have been considered in [8–17] with the focus primarily on critical exponents for finite time blow-up of solutions, and conditions for the existence of global solutions (see the review articles [18] and [19]). Moreover, for $p > 1$, solutions to (1) with two signed initial data have been considered in [20] and [21], whilst boundary value problems have been studied in [22] and [23].

The paper is structured as follows; in Section 2 we introduce the self-similar solution structure for [CP], and hence, determine an ordinary differential equation related to (1); the remainder of the paper concerns the study of particular solutions to this ordinary differential equation, which is re-written as an equivalent two-dimensional non-autonomous dynamical system. Specifically, in Section 3 we establish the existence of a two-parameter family of homoclinic connections on the equilibrium $(0, 0)$. Additionally, we determine bounds and estimates on the asymptotic approach of these solutions to $(0, 0)$. In Section 4, we establish the existence of a heteroclinic connection between the equilibrium points $(\pm(1 - p)^{1/(1-p)}, 0)$.

2. Self-similar structure

With $0 < p < 1$ and $T > 0$, we refer to $u : \mathbb{R} \times [0, T] \to \mathbb{R}$ as a solution to [CP] when u satisfies (1)–(2) with regularity,

$$u \in L^\infty(\mathbb{R} \times [0, T]) \cap C(\mathbb{R} \times [0, T]) \cap C^{2,1}(\mathbb{R} \times (0, T]). \tag{3}$$

Observe that $u^\pm : \mathbb{R} \times [0, T] \to \mathbb{R}$ given by

$$u^\pm(x, t) = \pm((1 - p)t)^{1/(1-p)} \quad \forall (x, t) \in \mathbb{R} \times [0, T]$$