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On horizontal Hardy, Rellich, Caffarelli–Kohn–Nirenberg and p-sub-Laplacian inequalities on stratified groups [☆]

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Abstract

In this paper, we present a version of horizontal weighted Hardy–Rellich type and Caffarelli–Kohn–Nirenberg type inequalities on stratified groups and study some of their consequences. Our results reflect on many results previously known in special cases. Moreover, a new simple proof of the Badiale–Tarantello conjecture [2] on the best constant of a Hardy type inequality is provided. We also show a family of Poincaré inequalities as well as inequalities involving the weighted and unweighted p-sub-Laplacians.

MSC: 22E30; 43A80

Keywords: Hardy inequality; Rellich inequality; Caffarelli–Kohn–Nirenberg inequality; p-sub-Laplacian; Horizontal estimate; Stratified group

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1. Introduction

Consider the following inequality

$$\left\| \frac{f(x)}{\|x\|} \right\|_{L^p(\mathbb{R}^n)} \leq \frac{p}{n-p} \|\nabla f\|_{L^p(\mathbb{R}^n)}, \quad 1 \leq p < n, \tag{1.1}$$

where ∇ is the standard gradient in $\mathbb{R}^n$, $f \in C_0^\infty(\mathbb{R}^n \backslash \{0\})$, $\|x\| = \sqrt{x_1^2 + ... + x_n^2}$, and the constant $\frac{p}{n-p}$ is known to be sharp. The one-dimensional version of (1.1) for $p = 2$ was first discovered by Hardy in [29], and then for other p in [30], see also [30] for the story behind these inequalities. Since then the inequality (1.1) has been widely analysed in many different settings (see e.g. [1–10,12,13,16,17,19,21,31,32]). Nowadays there is vast literature on this subject, for example, the MathSciNet search shows about 5000 research works related to this topic. On homogeneous Carnot groups (or stratified groups) inequalities of this type have been also intensively investigated (see e.g. [14,26–28,34–37,39]). In this case inequality (1.1) takes the form

$$\left\| \frac{f(x)}{d(x)} \right\|_{L^p(\mathbb{G})} \leq \frac{p}{Q-p} \|\nabla_H f\|_{L^p(\mathbb{G})}, \quad Q \geq 3,\ 1 < p < Q, \tag{1.2}$$

where Q is the homogeneous dimension of the stratified group $\mathbb{G}$, ∇_H is the horizontal gradient, and $d(x)$ is the so-called $\mathcal{L}$-gauge, which is a particular homogeneous quasi-norm obtained from the fundamental solution of the sub-Laplacian, that is, $d(x)^{2-Q}$ is a constant multiple of Folland's [23] (see also [24]) fundamental solution of the sub-Laplacian on $\mathbb{G}$. For a short review in this direction and some further discussions we refer to our recent papers [41–45] and [40] as well as to references therein.

The main aim of this paper is to give analogues of Hardy type inequalities on stratified groups with horizontal gradients and weights. Actually we obtain more than that, i.e., we prove general (horizontal) weighted Hardy, Rellich and Caffarelli–Kohn–Nirenberg type inequalities on stratified groups. Our results extend known Hardy type inequalities on abelian and Heisenberg groups, for example (see e.g. [2] and [11]). For the convenience of the reader let us now briefly recapture the main results of this paper. Let $\mathbb{G}$ be a homogeneous stratified group of homogeneous dimension Q, and let $X_1, \dots, X_N$ be left-invariant vector fields giving the first stratum of the Lie algebra of $\mathbb{G}$, $\nabla_H = (X_1, \dots, X_N)$, with the sub-Laplacian

$$\mathcal{L} = \sum_{k=1}^{N} X_k^2.$$

Denote the variables on $\mathbb{G}$ by $x = (x', x'') \in \mathbb{G}$, where x' corresponds to the first stratum. For precise definitions we refer to Section 2.

Thus, to summarise briefly, in this paper we establish the following results:

- **(Hardy inequalities)** Let $\mathbb{G}$ be a stratified group with N being the dimension of the first stratum, and let $\alpha, \beta \in \mathbb{R}$. Then for all complex-valued functions $f \in C_0^\infty(\mathbb{G} \backslash \{x' = 0\})$ and $1 < p < \infty$, we have the following L^p-Caffarelli–Kohn–Nirenberg type inequality