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The regularized 3D Boussinesq equations with fractional Laplacian and no diffusion

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Abstract

In this paper, we study the 3D regularized Boussinesq equations. The velocity equation is regularized à la Leray through a smoothing kernel of order α in the nonlinear term and a β-fractional Laplacian; we consider the critical case $\alpha+\beta=\frac{5}{4}$ and we assume $\frac{1}{2}<\beta<\frac{5}{4}$. The temperature equation is a pure transport equation, where the transport velocity is regularized through the same smoothing kernel of order α. We prove global well posedness when the initial velocity is in H^r and the initial temperature is in $H^{r-\beta}$ for $r>\max(2\beta,\beta+1)$. This regularity is enough to prove uniqueness of solutions. We also prove a continuous dependence of solutions on the initial conditions.

MSC: primary 35Q35, 76D03; secondary 35Q86

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1. Introduction

We consider the Boussinesq system in a d-dimensional space:

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$$\begin{cases} \partial_t v + (v \cdot \nabla) v - \nu \Delta v + \nabla p = \theta e_d \\ \partial_t \theta + v \cdot \nabla \theta = 0 \\ \nabla \cdot v = 0 \end{cases} \tag{1}$$

where $v = v(t, x)$ denotes the velocity vector field, $p = p(t, x)$ the scalar pressure and $\theta = \theta(t, x)$ a scalar quantity, which can represent either the temperature of the fluid or the concentration of a chemical component; e_d is the unit vector $(0, \dots, 0, 1)$, the viscosity ν is a positive constant. Suitable initial conditions v_0, θ_0 and boundary conditions (if needed) are given.

For $d = 2$, the well posedness of system (1) in the plane has been studied by several authors under different assumptions on the initial data (see [12,7,1,11,8,9]). For $d = 3$, very little is known; it has been proven that there exists a local smooth solution. Some regularity criterions to get a global (in time) solution have been obtained in [21,10]. Otherwise, in the particular case of axisymmetric initial data, [2] shows the global well posedness for the Boussinesq system in the whole space.

To overcome the difficulties of the three-dimensional case, different models have been proposed. For instance, one can regularize the equation for the velocity by putting a fractional power of the Laplacian; this hyper-dissipative Boussinesq system takes the form

$$\begin{cases} \partial_t v + (v \cdot \nabla) v + \nu (-\Delta)^\beta v + \nabla p = \theta e_3 \\ \partial_t \theta + v \cdot \nabla \theta = 0 \\ \nabla \cdot v = 0 \end{cases} \tag{2}$$

For $\beta > \frac{5}{4}$, [28] proved the global well posedness. This result has been improved by Ye [27], allowing $\beta = \frac{5}{4}$.

Notice that for zero initial temperature θ_0, the Boussinesq system reduces to the Navier–Stokes equations. It is well known that the three-dimensional Navier–Stokes equations have either a unique local smooth solution or a global weak solution. The questions related to the local smooth solution being global or the global weak solution being unique are very challenging problems that are still open since the seminal work of Leray. For this reason, modifications of different types have been considered for the three-dimensional Navier–Stokes equations. On one side there is the hyper-viscous model, i.e. (2) with zero initial temperature; when $\beta \ge \frac{5}{4}$, uniqueness of the weak solutions has been proved in [17] (see Remark 6.11 of Chapter 1) and [18]. On the other hand, Olson and Titi in [20] suggested to regularize the equations by modifying two terms. For a particular model of fluid dynamics, they replaced the dissipative term by a fractional power of the Laplacian and they regularized the bilinear term of vorticity stretching à la Leray. The well posedness of those equations is obtained by asking a balance between the modification of the nonlinearity and of the viscous dissipation; at least one of them has to be strong enough, while the other might be weak. Similarly, Barbato, Morandin and Romito in [4] considered the Leray-α Navier–Stokes equations with fractional dissipation

$$\begin{cases} \partial_t v + (u \cdot \nabla) v + \nu (-\Delta)^\beta v + \nabla p = 0 \\ v = u + (-\Delta)^\alpha u \\ \nabla \cdot u = \nabla \cdot v = 0 \end{cases} \tag{3}$$

and proved that this system is well posed when $\alpha + \beta \ge \frac{5}{4}$ (with $\alpha, \beta \ge 0$); even some logarithmic corrections can be included, but we do not specify this detail, since it is not related to our