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Configurations of periodic orbits for equations with delayed positive feedback

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Dedicated to Professor Tibor Krisztin on the occasion of his 60th birthday

Abstract

We consider scalar delay differential equations of the form

$$\dot{x}(t) = -\mu x(t) + f(x(t-1)),$$

where $\mu > 0$ and f is a nondecreasing C^1-function. If χ is a fixed point of $f_\mu : \mathbb{R} \ni u \mapsto f(u)/\mu \in \mathbb{R}$ with $f'_\mu(\chi) > 1$, then $[-1, 0] \ni s \mapsto \chi \in \mathbb{R}$ is an unstable equilibrium. A periodic solution is said to have large amplitude if it oscillates about at least two fixed points $\chi_- < \chi_+$ of f_μ with $f'_\mu(\chi_-) > 1$ and $f'_\mu(\chi_+) > 1$. We investigate what type of large-amplitude periodic solutions may exist at the same time when the number of such fixed points (and hence the number of unstable equilibria) is an arbitrary integer $N \geq 2$. It is shown that the number of different configurations equals the number of ways in which N symbols can be parenthesized. The location of the Floquet multipliers of the corresponding periodic orbits is also discussed.

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Keywords: Delay differential equation; Positive feedback; Large-amplitude periodic solution; Slow oscillation; Floquet multiplier

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1. Introduction

We study the delay differential equation

$$\dot{x}(t) = -\mu x(t) + f(x(t-1)) \tag{1.1}$$

under the hypotheses

$(H0)$ $\mu > 0$,
$(H1)$ feedback function $f \in C^1(\mathbb{R}, \mathbb{R})$ is nondecreasing.

If $\chi \in \mathbb{R}$ is a fixed point of $f_\mu : \mathbb{R} \ni u \mapsto f(u)/\mu \in \mathbb{R}$, then $\hat{\chi} \in C = C([-1,0], \mathbb{R})$, defined by $\hat{\chi}(s) = \chi$ for all $s \in [-1, 0]$, is an equilibrium of the semiflow. In this paper we assume that

$(H2)$ if χ is a fixed point of f_μ, then $f'_\mu(\chi) \neq 1$.

This hypothesis guarantees that all equilibria are hyperbolic. It is well known that if χ is an unstable fixed point of f_μ (that is, if $f'_\mu(\chi) > 1$), then $\hat{\chi}$ is an unstable equilibrium. If χ is a stable fixed point of f_μ (that is, if $f'_\mu(\chi) < 1$), then $\hat{\chi}$ is also stable (exponentially stable). The stable and unstable equilibria alternate in pointwise ordering.

Mallet-Paret and Sell have verified a Poincaré–Bendixson type result for (1.1) in the case when $f'(u) > 0$ for all $u \in \mathbb{R}$ [17]. Krisztin, Walther and Wu obtained further detailed results on the structure of the solutions (see e.g. [9,7,8,12–14]). They have characterized the geometrical and topological properties of the closure of the unstable set of an unstable equilibrium, the so-called Krisztin–Walther–Wu attractor. If there is only one unstable equilibrium, sufficient conditions can be given for the closure of the unstable set to be the global attractor.

The chief motivation for the present work comes from the paper [17] of Mallet-Paret and Sell. They have shown that if $f'(u) > 0$ for all $u \in \mathbb{R}$, then

$$\pi_2 : C \ni \varphi \mapsto (\varphi(0), \varphi(-1)) \in \mathbb{R}^2$$

maps different (nonconstant and constant) periodic orbits of (1.1) onto disjoint sets in $\mathbb{R}^2$, and the images of nonconstant periodic orbits are simple closed curves in $\mathbb{R}^2$. They have also shown that a nonconstant periodic solution $p : \mathbb{R} \to \mathbb{R}$ of (1.1) oscillates about a fixed point χ of f_μ if and only if $\pi_2 \hat{\chi} = (\chi, \chi)$ is in the interior of $\pi_2 \{p_t : t \in \mathbb{R}\}$. See Fig. 1.1. These results give a strong restriction on what type of periodic solutions the equation may have for the same feedback function f: Suppose that $p^1 : \mathbb{R} \to \mathbb{R}$ and $p^2 : \mathbb{R} \to \mathbb{R}$ are periodic solutions of equation (1.1). For both $i \in \{1, 2\}$, let E_i be the set of those fixed points of f_μ about which p^i oscillates. Then either $E_1 \subseteq E_2$ or $E_2 \subseteq E_1$ or $E_1 \cap E_2 = \emptyset$. We can easily extend these assertions to the case when $f'(u) \geq 0$ for all $u \in \mathbb{R}$, see Proposition 3.4 in Section 3.

This paper considers large-amplitude periodic solutions: periodic solutions oscillating about at least two unstable fixed points of f_μ. Fig. 1.2 lists all configurations of large-amplitude periodic solutions allowed by the previously cited results of Mallet-Paret and Sell in case there are three and four unstable equilibria, respectively. It is a natural question whether all of them indeed exist for some nonlinearities f.

Allowing any number of unstable equilibria, we confirm the existence of all possible configurations of large-amplitude periodic solutions by constructing the suitable feedback functions and