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The estimate of the amplitude of limit cycles of symmetric Liénard systems

Yuli Cao, Changjian Liu *

Department of Mathematics, Soochow University, Suzhou, 215006, China

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Abstract

Symmetric Liénard system $\dot{x} = y - F(x)$, $\dot{y} = -g(x)$ (i.e. $F(x)$ and $g(x)$ are odd functions) is studied. It is well known that under some hypotheses, this system has a unique limit cycle. We develop a method to give both the upper bound and lower bound of the amplitude, which is the maximal value of the x-coordinate, of the unique limit cycle. As an application, we consider van der Pol equation $\dot{x} = y - \mu(x^3/3 - x)$, $\dot{y} = -x$, where $\mu > 0$. Denote by $A(\mu)$ the amplitude of its unique limit cycle, then for any μ, we show that $A(\mu) < 2.0976$ and for $\mu = 1, 2$, we show that $A(\mu) > 2$. Both the upper bound and the lower bound improve the existing ones.

MSC: 34C07; 34C05; 34C14

Keywords: Liénard system; Limit cycles; Van der Pol equation

1. Introduction and main results

Consider Liénard system which has the form

$$\dot{x} = y - F(x), \qquad \dot{y} = -g(x) \tag{1}$$

* Corresponding author.
E-mail addresses: 20144207005@stu.suda.edu.cn (Y. Cao), liucj@suda.edu.cn (C. Liu).

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or equivalent

$$\dot{x} = v, \qquad \dot{y} = -g(x) - f(x)v, \tag{2}$$

or equivalent

$$\ddot{x} + f(x)\dot{x} + g(x) = 0, \tag{3}$$

where $F(x) = \int_0^x f(s)\mathrm{d}s$. Since system (1) appears in a lot of physical problems, the problem of limit cycles of system (1), such as the existence, uniqueness, number, relative position and location, has been widely studied, see the survey by J. Llibre [7] and two monographs, Y. Ye et al. [11] and Z. Zhang et al. [12]. Let us first recall the following well known result on the existence and uniqueness of limit cycles of system (1).

Theorem 1. *System (1) has a unique limit cycle and this limit cycle is global asymptotically stable, if the following hypotheses hold:*

1. $f(x)$ *is even and continuous in* $(-\infty, +\infty)$*;*
2. $F(x)$ *has a unique positive zero* $a > 0$*, and* $F(x)(x - a) > 0$ *for* $x > 0, x \neq a$*;*
3. $F(x)$ *is monotone increasing in* $(a, +\infty)$*, and* $F(x) \to +\infty$ *as* $x \to +\infty$*;*
4. $g(x)$ *is odd and continuous in* $(-\infty, +\infty)$*, and* $xg(x) > 0, \forall x \neq 0$.

The above hypotheses have obvious physical meanings, thus they become standard ones when one considers the existence and uniqueness of limit cycles of Liénard system. For the physical meaning of the hypotheses and the proof of Theorem 1, we recommend the readers the monograph [6] or [4] or [9] or [5]. In this paper, we suppose that all these four hypotheses hold so that system (1) has a unique limit cycle and we are concerned with the location of this limit cycle, or concretely, the amplitude x^M of this limit cycle. This amplitude of a limit cycle is defined as the maximal value of the x-coordinate on the limit cycle and physically it is the maximal deviation of the oscillation from the equilibrium state for system (2).

From the proof of Theorem 1, it is easy to see that the amplitude $x^M > a$ (the proof is also shown in the next section), but to our knowledge, there is little estimate on the lower bound of the amplitude x^M, in [3], the authors find a method to give some estimate of the lower bound of the amplitude x^M. On the other hand, there are several papers, such as [1,8,2], which can give upper bounds of x^M. Since their estimates are complicated, we do not list them here. Recently, in [10] Lijun Yang and Xianwu Zeng proved the following theorem, which can give an explicit upper bound of the amplitude of the unique limit cycle in Theorem 1 under some additional conditions.

Theorem 2. *In addition to the hypotheses of Theorem 1, assume that*

(1) $f(x)$ *has a unique positive zero* $a_1 > 0$ *and* $f(x)(x - a_1) > 0$ *for* $x > 0, x \neq a_1$*;*
(2) $f(x)/g(x)$ *is monotone increasing on* $(a_1, +\infty)$*;*
(3) $\int_0^x F(s)g(s)\mathrm{d}s > 0$ *for sufficiently large* $x > 0$.

Then the unique limit cycle in Theorem 1 locates in the strip region $|x| < \bar{x}^*$*, where* $\bar{x}^* > 0$ *is uniquely determined by* $\int_0^{\bar{x}^*} F(s)g(s)\mathrm{d}x = 0$*. In other words,* $x^M < \bar{x}^*$.