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Inverse scattering on the line for the matrix Sturm–Liouville equation

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Abstract

The inverse scattering problem is studied for the matrix Sturm–Liouville equation on the line. Necessary and sufficient conditions for the scattering data are obtained.

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Keywords: Matrix Sturm–Liouville equation; Inverse scattering problem; Necessary and sufficient conditions; Gelfand–Levitan–Marchenko equation

1. Introduction

In this paper, we consider the matrix Sturm–Liouville (also called Schrödinger) equation on the real line:

$$-Y'' + Q(x)Y = \rho^2 Y, \quad -\infty < x < \infty. \tag{1.1}$$

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Here $Y = [y_k(x)]_{k=1}^m$ is a vector function, ρ is the spectral parameter, and $Q(x) = [Q_{jk}(x)]_{j,k=1}^n$ is the self-adjoint matrix potential ($Q(x) = Q^*(x)$), satisfying the condition

$$\int_{-\infty}^{\infty} (1 + |x|)|Q_{jk}(x)|\, dx < \infty, \quad j, k = \overline{1, m}. \tag{1.2}$$

The inverse scattering problem is studied, which consists in recovering of the potential Q from the given scattering data.

There is an extensive literature on the inverse spectral and scattering problems (see monographs [1–5] and references therein). Such problems arise in quantum mechanics, geophysics, electronics, chemistry and other branches of science and engineering. One of the most important applications is the inverse scattering method for integration of nonlinear evolution equations, such as Korteweg–de Vries (KdV) equation, nonlinear Schrödinger equation, sine-Gordon equation, Toda lattice, etc. (see [6–9]).

A complete analysis of the inverse scattering problem for the *scalar* Sturm–Liouville equation on the real line (equation (1.1) for $m = 1$) was carried out by L.D. Faddeev [10] and V.A. Marchenko [1]. They have obtained the characterization of the scattering data, in other words, necessary and sufficient conditions for the solvability of the inverse scattering problem.

Inverse scattering problems for *matrix* Sturm–Liouville operators appeared to be more difficult, than scalar ones, because of the more complicated structure of the discrete scattering data. For the matrix case, Z.S. Agranovich and V.A. Marchenko [11] solved the inverse scattering problem on the *half-line*, using the transformation operator method [1,2]. Later on, several authors studied the inverse scattering problem for the matrix Sturm–Liouville operator on the *line* [12–18]. In particular, M. Wadati and T. Kamijo [12] reduced the inverse problem for the self-adjoint potential to the Gelfand–Levitan–Marchenko equation (linear Fredholm integral equation, connecting the scattering data with the kernel of the transformation operator). E. Olmedilla [16] generalized their results to the non-self-adjoint case. F. Calogero and A. Degasperis [13] studied applications of the inverse scattering problem to the integration of matrix nonlinear evolution equations such as matrix KdV and Boomeron equations. However, as far as we know, a rigorous mathematical analysis of the solvability of the Gelfand–Levitan–Marchenko equation and characterization of the scattering data were not done before for the matrix case. The goal of this paper is to cover this gap, and provide necessary and sufficient conditions for the solvability of the inverse scattering problem for equation (1.1).

The paper is organized as follows. In Section 2, we introduce the left and the right scattering data for equation (1.1), study their properties, formulate the inverse scattering problem and give other preliminaries. In Section 3, the Gelfand–Levitan–Marchenko equation is derived. Although many of the results of Sections 2 and 3 appeared before in [11,12,20] and other literature, we provide their proofs for the convenience of the reader. In Section 4, we give sufficient conditions for the unique solvability of the Gelfand–Levitan–Marchenko equation. However, these conditions are not sufficient for the solvability of the inverse scattering problem, because the solution of the main equation, constructed, for example, by the right scattering data, is guaranteed to fulfill (1.2) only on the right half-line. Therefore we have to study the connection between the left and the right scattering data. In Section 5, we formulate and prove our main result, Theorem 5.3 on the necessary and sufficient conditions on the scattering data of the matrix Sturm–Liouville operator. We show how the left scattering data can be constructed by the right ones. Then the resulting potential satisfies (1.2) on the full line, and we prove that its scattering data coincide with the given

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