Available online at www.sciencedirect.com

ScienceDirect

J. Differential Equations 262 (2017) 2254–2285

Journal of Differential Equations

www.elsevier.com/locate/jde

Dependence of solutions of nonsmooth differential-algebraic equations on parameters

Peter G. Stechlinski, Paul I. Barton *

Process Systems Engineering Laboratory, Massachusetts Institute of Technology, Cambridge, MA 02139, United States

Received 17 June 2016; revised 26 October 2016

Available online 10 November 2016

Abstract

The well-posedness of nonsmooth differential-algebraic equations (DAEs) is investigated. More specifically, semi-explicit DAEs with Carathéodory-style assumptions on the differential right-hand side functions and local Lipschitz continuity assumptions on the algebraic equations. The DAEs are classified as having differential index one in a generalized sense; solution regularity is formulated in terms of projections of generalized (Clarke) Jacobians. Existence of solutions is derived under consistency and regularity of the initial data. Uniqueness of a solution is guaranteed under analogous Carathéodory ordinary-differential equation uniqueness assumptions. The continuation of solutions is established and sufficient conditions for continuous and Lipschitzian parametric dependence of solutions are also provided. To accomplish these results, a theoretical tool for analyzing nonsmooth DAEs is provided in the form of an extended nonsmooth implicit function theorem. The findings here are a natural extension of classical results and lay the foundation for further theoretical and computational analyses of nonsmooth DAEs.

MSC: 34A09; 34A12; 26B10

Keywords: Semi-explicit; Generalized derivatives; Implicit function theorems; Existence; Uniqueness; Consistent initialization

* Corresponding author.
E-mail addresses: pstechli@mit.edu (P.G. Stechlinski), pib@mit.edu (P.I. Barton).
URL: http://yoric.mit.edu (P.I. Barton).

http://dx.doi.org/10.1016/j.jde.2016.10.041

1. Introduction

The focus of this article is to analyze the well-posedness of Carathéodory semi-explicit DAE systems. Namely, the following initial-value problem (IVP) in semi-explicit DAEs:

$$\dot{\mathbf{x}}(t,\mathbf{p}) = \mathbf{f}(t,\mathbf{p},\mathbf{x}(t,\mathbf{p}),\mathbf{y}(t,\mathbf{p})), \tag{1a}$$

$$\mathbf{0}_{n_y} = \mathbf{g}(t,\mathbf{p},\mathbf{x}(t,\mathbf{p}),\mathbf{y}(t,\mathbf{p})), \tag{1b}$$

$$\mathbf{x}(t_0,\mathbf{p}) = \mathbf{f}_0(\mathbf{p}), \tag{1c}$$

in which t is the independent variable; $\mathbf{p}$ is a vector of the problem parameters; $\mathbf{x}$ are the differential state variables; $\mathbf{y}$ are the algebraic state variables; and the right-hand side functions $\mathbf{f}: D \to \mathbb{R}^{n_x}$ and $\mathbf{g}: D \to \mathbb{R}^{n_y}$ are nonsmooth mappings. Here $D \subset \mathbb{R} \times \mathbb{R}^{n_p} \times \mathbb{R}^{n_x} \times \mathbb{R}^{n_y}$ is open and connected where n_p, n_x, n_y are positive integers. The function $\mathbf{f}_0$, which maps the projection $\pi_p D$ (i.e., D onto $\mathbb{R}^{n_p}$) to the projection $\pi_x D$, may also be nonsmooth.

Example 1.1. Consider the following IVP in semi-explicit DAEs:

$$\dot{x}(t,p) = \mathrm{sign}(t-0.5) + (1.5|1-\eta_y|^{\frac{1}{3}} - 1)\mathrm{H}(t-1), \tag{2a}$$

$$0 = |x(t,p)| + |y(t,p)| - 1, \tag{2b}$$

$$x(t_0,p) = \min\{0,p\}, \tag{2c}$$

where sign(·) and H(·) denote the signum and Heaviside functions, respectively. Suppose that $t_0 := 0$ is prescribed and consider the following mappings:

$$\mathbf{z}^\dagger \equiv (x^\dagger, y^\dagger) : [0,1] \times \{-0.5\} \to \mathbb{R}^2 : (t,p) \mapsto \begin{cases} (-t-0.5, 0.5-t), & \text{if } t \in [0,0.5], \\ (t-1.5, t-0.5), & \text{if } t \in (0.5,1], \end{cases}$$

and

$$\mathbf{z}_\dagger \equiv (x_\dagger, y_\dagger) : [0,1] \times \{-0.5\} \to \mathbb{R}^2 : (t,p) \mapsto \begin{cases} (-t-0.5, 0.5-t), & \text{if } t \in [0,0.5], \\ (t-1.5, 0.5-t), & \text{if } t \in (0.5,1]. \end{cases}$$

The mappings $\mathbf{z}^\dagger$ and $\mathbf{z}_\dagger$ both pass through the point $(t_0, \mathbf{p}_0, \mathbf{x}_0, \mathbf{y}_0) := (0, -0.5, -0.5, 0.5)$. Observe that $\mathbf{z}^\dagger(\cdot, -0.5)$ and $\mathbf{z}_\dagger(\cdot, -0.5)$ both satisfy (2a) for almost every $t \in [0,1]$, (2b) for all $t \in [0,1]$, and (2c) at $t = t_0$. See Fig. 1 for an illustration.

DAEs provide a natural framework for the dynamic modeling and simulation of a wide range of engineering applications found in network modeling, mechanical multibody systems, constrained variational problems, and fluid dynamics (see [1,2] and the references therein). Nonsmoothness is an inherent feature of dynamic models of chemical processes [3]. For example, sources of nonsmoothness in campaign continuous pharmaceutical manufacturing include thermodynamic phase changes (e.g., flash evaporation, liquid-liquid extraction), flow transitions (e.g., laminar-turbulent-choked transitions), flow control devices (e.g., nonreturn valves, weirs), crystallization kinetics that vary whether the solution is supersaturated or unsaturated, etc. [4–6].