ARTICLE IN PRESS YJDEQ:8592

Available online at www.sciencedirect.com

ScienceDirect

J. Differential Equations ••• (••••) •••–•••

Journal of Differential Equations

www.elsevier.com/locate/jde

Positive solutions of a one-dimensional indefinite capillarity-type problem: a variational approach ☆

Julián López-Gómez [a], Pierpaolo Omari [b,*], Sabrina Rivetti [b]

[a] Universidad Complutense de Madrid, Departamento de Matemática Aplicada, Plaza de Ciencias 3, 28040 Madrid, Spain

[b] Università degli Studi di Trieste, Dipartimento di Matematica e Geoscienze, Sezione di Matematica e Informatica, Via A. Valerio 12/1, 34127 Trieste, Italy

Received 29 July 2016

Abstract

We prove the existence and the multiplicity of positive solutions of the one-dimensional capillarity-type problem

$$-\left(u'/\sqrt{1+(u')^2}\right)' = a(x)f(u), \quad u'(0)=0,\ u'(1)=0,$$

where $a \in L^1(0,1)$ changes sign and $f : [0,+\infty) \to [0,+\infty)$ is continuous and has a power-like behavior at the origin and at infinity. Our approach is variational and relies on a regularization procedure that yields bounded variation solutions which are of class $W^{2,1}_{\mathrm{loc}}$, and hence satisfy the equation pointwise almost everywhere, on each open interval where the weight function a has a constant sign.

MSC: primary 35J93, 34B18; secondary 34B15, 35B65, 35D10, 35J20, 49Q20, 53A10, 76D45

☆ This paper was written under the auspices of INdAM-GNAMPA. The first and the second named authors have been supported by "The Spanish Ministry of Economy and Competitiveness of Spain under Research Grants MTM2012-30669 and MTM2015-65899-P". The second named author has been also supported by "Università degli Studi di Trieste – Finanziamento di Ateneo per Progetti di Ricerca Scientifica – FRA 2015".

* Corresponding author.

E-mail addresses: julian@mat.ucm.es (J. López-Gómez), omari@units.it (P. Omari), sabrina.rivetti@gmail.com (S. Rivetti).

http://dx.doi.org/10.1016/j.jde.2016.10.046
0022-0396/

Please cite this article in press as: J. López-Gómez et al., Positive solutions of a one-dimensional indefinite capillarity-type problem: a variational approach, J. Differential Equations (2016), http://dx.doi.org/10.1016/j.jde.2016.10.046

Keywords: Prescribed curvature equation; Neumann boundary condition; Indefinite weight; Bounded variation solution; Regularization procedure; Variational method

1. Introduction and statements

In this paper we are interested in the existence of positive solutions of the quasilinear Neumann problem

$$\begin{cases} -\left(\dfrac{u'}{\sqrt{1+(u')^2}}\right)' = a(x)f(u) \quad \text{in } (0,1), \\ u'(0)=0, \ u'(1)=0, \end{cases} \tag{1.1}$$

where $a \in L^1(0,1)$ changes sign and $f : [0,+\infty) \to [0,+\infty)$ is a continuous function having superlinear, or sublinear, growth at 0 and at $+\infty$.

Problem (1.1) is a particular, one-dimensional, version of the elliptic problem

$$\begin{cases} -\mathrm{div}\left(\dfrac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = g(x,u) \quad \text{in } \Omega, \\ -\dfrac{\nabla u \cdot \nu}{\sqrt{1+|\nabla u|^2}} = \sigma \quad \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

where Ω is a bounded regular domain in $\mathbb{R}^N$, with outward pointing normal ν, and $g : \Omega \times \mathbb{R} \to \mathbb{R}$ and $\sigma : \partial\Omega \to \mathbb{R}$ are given functions. This problem plays a relevant role in the mathematical analysis of a number of physical or geometrical issues, such as capillarity phenomena for incompressible fluids, reaction–diffusion processes where the flux features saturation at high regimes, or prescribed mean curvature problems for cartesian surfaces in the Euclidean space. Significant references include [41,58,10,22,31,28,34,32,30,36,39,33,40,16].

Although there is a large amount of literature devoted to the existence of positive solutions for semilinear elliptic problems with superlinear indefinite nonlinearities, starting with [7,1,2,9,8,3], no result is available for the problem (1.2), even in the one-dimensional case (1.1), in spite of the interest that this topic may have both mathematically and from the point of view of the applications.

As it will become clear later, according to Proposition 1.1 below, the existence of a positive solution for the homogeneous Neumann problem (1.1) forces the right hand side of the equation to change sign, thus ruling out the possibility, if f is non-negative, that the sign of the weight function a be constant. Hence, the absence of any previous result in the existing literature might be attributable to the fact that superlinear indefinite weighted problems are fraught with a number of technical difficulties which do not arise in dealing with purely sublinear or superlinear problems, even in the most classical semilinear case, not to talk about the degenerate quasilinear problem dealt with in this paper. In addition, as an effect of the spatial heterogeneities incorporated into the formulation of the problem the complexity of the structure of the solution sets might be quite intricate, even in the semilinear case [35,48,47,46,13,14]. This is an extremely challenging problem in the context of (1.1), which will be addressed elsewhere (see, e.g., [45]).

Please cite this article in press as: J. López-Gómez et al., Positive solutions of a one-dimensional indefinite capillarity-type problem: a variational approach, J. Differential Equations (2016), http://dx.doi.org/10.1016/j.jde.2016.10.046