Available online at www.sciencedirect.com
ScienceDirect
J. Differential Equations ••• (••••) •••–•••
Journal of Differential Equations
www.elsevier.com/locate/jde

On some fractional equations with convex–concave and logistic-type nonlinearities

Giulia Carboni, Dimitri Mugnai [*,1]

Dipartimento di Matematica e Informatica, Università di Perugia, Via Vanvitelli 1, 06123, Perugia – Italy

Received 14 August 2016; revised 2 October 2016

Abstract

We consider existence and multiplicity results for a semilinear problem driven by the square root of the negative Laplacian in presence of a nonlinear term which is the difference of two powers. In the case of convex–concave powers, a precise description of the problem at the threshold value of a given parameter is established through variational methods and truncation techniques.

MSC: 35R11; 35J20; 35J65; 35J61

Keywords: Fractional Laplacian; Convex and concave nonlinearities; Threshold value

1. Introduction and main results

In this paper we are concerned with the existence and the multiplicity of solutions for problems whose prototype is

[*] Corresponding author. Fax: +39 075 5855024.
E-mail addresses: giulia.carboni@unipg.it (G. Carboni), dimitri.mugnai@unipg.it (D. Mugnai).
[1] D.M. is a member of the INDAM Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni (GNAMPA). Research supported by the INdAM-GNAMPA Project 2016 *Nonlocal and quasilinear operators in presence of singularities*.

http://dx.doi.org/10.1016/j.jde.2016.10.045
0022-0396/© 2016 Elsevier Inc. All rights reserved.

$$(P_\lambda) \qquad \begin{cases} \sqrt{-\Delta} u = \lambda |u|^{q-2}u - h(x)|u|^{p-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

More generally, we are concerned with problems of the form

$$(Q_\lambda) \qquad \begin{cases} \sqrt{-\Delta} u = \lambda |u|^{q-2}u - h(x)g(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Here Ω is a bounded domain of $\mathbb{R}^N$ ($N \geq 1$) and $1 < q < p < 2^\#$, where

$$2^\# = \begin{cases} \infty & \text{if } N = 1, \\ \frac{2N}{N-1} & \text{if } N \geq 2 \end{cases}$$

is the Sobolev fractional critical exponent, $h : \Omega \to \mathbb{R}$ is a continuous and positive function, g is a Carathéodory function such that $g(x,t) \sim |t|^{p-2}t$, and $\sqrt{-\Delta}$ denotes the square root of the negative Laplacian. Such an operator arises in several branches of Physics, Probability, Biology and Finance, and it is the infinitesimal generator of a stable Lévy process (see, for instance, [6–8, 11,13–18,20,22,24,25]).

If $2 < q < p$, problem (P_λ) is a *convex–concave* elliptic problem involving the fractional Laplacian. Local problems with this type of nonlinearities were widely studied. A first paper is the seminal one by Ambrosetti–Brezis–Cerami [2], where they considered the *concave–convex* Dirichlet problem

$$\begin{cases} -\Delta u = \lambda u^{q-1} + u^{p-1} & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where $1 < q < 2 < p < 2^*$ ($2^* = \frac{2N}{N-2}$ if $N \geq 3$, $2^* = \infty$ if $N = 1, 2$). They proved that there exists $\lambda_0 > 0$ such that problem (1) has at least two solutions for all $\lambda \in (0, \lambda_0)$, one solution for $\lambda = \lambda_0$, and no solutions for $\lambda > \lambda_0$.

Of course, the nonlinearity in (1) has a different nature from the one in (P_λ). On the other hand, a related problem with convex–concave nonlinearities was considered by Alama–Tarantello in [1], where the following Dirichlet problem was treated:

$$\begin{cases} -\Delta u = \lambda u + k(x)u^{q-1} - h(x)u^{p-1} & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{su } \partial\Omega. \end{cases}$$

Here, $2 < q < p$, h ad k are nonnegative *bounded* functions, and the qualitative study is based on integrability properties of the quotient k^p / h^q.

In their recent paper [28], Rădulescu and Repovš developed the previous ideas considering the *concave–concave* Dirichlet problem

Please cite this article in press as: G. Carboni, D. Mugnai, On some fractional equations with convex–concave and logistic-type nonlinearities, J. Differential Equations (2016), http://dx.doi.org/10.1016/j.jde.2016.10.045