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Non-divergence parabolic equations of second order with critical drift in Lebesgue spaces

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Abstract

We consider uniformly parabolic equations and inequalities of second order in the non-divergence form with drift

$$-u_t + Lu = -u_t + \sum_{ij} a_{ij} D_{ij} u + \sum b_i D_i u = 0 \, (\geq 0, \, \leq 0)$$

in some domain $Q \subset \mathbb{R}^{n+1}$. We prove growth theorems and the interior Harnack inequality as the main results. In this paper, we will only assume the drift b is in certain Lebesgue spaces which are critical under the parabolic scaling but not necessarily to be bounded. In the last section, some applications of the interior Harnack inequality are presented. In particular, we show there is a "universal" spectral gap for the associated elliptic operator. The counterpart for uniformly elliptic equations of second order in non-divergence form is shown in [19].

Keywords: Second-order parabolic equations; Harnack inequality; Measurable coefficients; Spectral gap

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1. Introduction

1.1. General introduction

The qualitative properties of solutions to partial differential equations have been intensively studied for a long time. In this paper, we consider the qualitative properties of solutions to the uniformly parabolic equation in non-divergence form,

$$-u_t + Lu := -u_t + \sum_{ij} a_{ij} D_{ij} u + \sum_i b_i D_i u = 0 \tag{1.1}$$

and the associated inequalities: $-u_t + Lu \geq 0$ and $-u_t + Lu \leq 0$. Throughout the paper, we use the notations $D_i := \frac{\partial}{\partial x_i}$, $D_{ij} := \frac{\partial^2}{\partial x_i \partial x_j}$ and $u_t := \frac{\partial u}{\partial t}$. We assume $b = (b_1, \ldots, b_n)$ and a_{ij}'s are real measurable, a_{ij}'s also satisfy the *uniform parabolicity condition*

$$\forall \xi \in \mathbb{R}^n,\ \nu^{-1} |\xi|^2 \leq \sum_{i,j=1}^{n} a_{ij}(X) \xi_i \xi_j, \quad \sum_{i,j=1}^{n} a_{ij}^2 \leq \nu^2 \tag{1.2}$$

with some constant $\nu \geq 1$, $\forall X = (x, t)$ in the domain of definition $Q \subset \mathbb{R}^{n+1}$.

For the drift b, we will only require it is in certain Lebesgue spaces which are critical under the parabolic scaling. To formulate our setting more precisely, we assume over the domain of definition Q,

$$\|b\|_{L_x^p L_t^q} := \left(\int \left[\int |b(x,t)|^q \, dt \right]^{\frac{p}{q}} dx \right)^{\frac{1}{p}} =: S(Q) < \infty, \tag{1.3}$$

for some constants p, $q \geq 1$ such that

$$\frac{n}{p} + \frac{2}{q} = 1. \tag{1.4}$$

By "critical", we mean that with the $L_x^p L_t^q$ norm, the drift is scaling invariant under the parabolic scaling: for $r > 0$,

$$x \to r^{-1} x,\ t \to r^{-2} t.$$

Indeed, suppose u satisfies

$$-u_t + \sum_{ij} a_{ij} D_{ij} u + \sum_i b_i D_i u = 0,$$

in a domain $Q \in \mathbb{R}^{n+1}$. Then for any constant $r > 0$, let

$$\tilde{x} = r^{-1} x,\ \tilde{t} = r^{-2} t.$$