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The Fadell–Rabinowitz index and multiplicity of non-contractible closed geodesics on Finsler $\mathbb{R}P^n$

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Abstract

In this paper, we prove that for every irreversible Finsler n-dimensional real projective space $(\mathbb{R}P^n, F)$ with reversibility λ and flag curvature K satisfying $\frac{16}{9}\left(\frac{\lambda}{1+\lambda}\right)^2 < K \le 1$ with $\lambda < 3$, there exist at least $n-1$ non-contractible closed geodesics. In addition, if the metric F is bumpy with $\frac{64}{25}\left(\frac{\lambda}{1+\lambda}\right)^2 < K \le 1$ and $\lambda < \frac{5}{3}$, then there exist at least $2[\frac{n+1}{2}]$ non-contractible closed geodesics, which is the optimal lower bound due to Katok's example. The main ingredients of the proofs are the Fadell–Rabinowitz index theory of non-contractible closed geodesics on $(\mathbb{R}P^n, F)$ and the S^1-equivariant Poincaré series of the non-contractible component of the free loop space on $\mathbb{R}P^n$.

MSC: 53C22; 58E05; 58E10

Keywords: Real projective space; Non-contractible closed geodesics; Fadell–Rabinowitz index; Poincaré series; Multiplicity

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1. Introduction and main results

In this paper, we are interested in the existence of non-contractible closed geodesics on real projective space $\mathbb{R}P^n$ with an irreversible Finsler metric, which is the typically non-simply connected manifold with the fundamental group $\pi_1(\mathbb{R}P^n) = \mathbf{Z}_2$.

A closed curve on a Finsler manifold is a closed geodesic if it is locally the shortest path connecting any two nearby points on this curve. As usual, on any Finsler manifold (M, F), a closed geodesic $c : S^1 = \mathbf{R}/\mathbf{Z} \to M$ is *prime* if it is not a multiple covering (i.e., iteration) of any other closed geodesics. Here the m-th iteration c^m of c is defined by $c^m(t) = c(mt)$. The inverse curve c^{-1} of c is defined by $c^{-1}(t) = c(1-t)$ for $t \in \mathbf{R}$. Note that unlike Riemannian manifold, the inverse curve c^{-1} of a closed geodesic c on a irreversible Finsler manifold need not be a geodesic. We call two prime closed geodesics c and d *distinct* if there is no $\theta \in (0, 1)$ such that $c(t) = d(t + \theta)$ for all $t \in \mathbf{R}$. We shall omit the word *distinct* when we talk about more than one prime closed geodesic. For a closed geodesic c on (M, F), denote by P_c the linearized Poincaré map of c. Recall that a Finsler metric F is *bumpy* if all the closed geodesics on (M, F) are non-degenerate, i.e., $1 \notin \sigma(P_c)$ for any closed geodesic c.

There have been a great deal of works on the multiplicity of closed geodesics for simply connected manifolds (cf. [1,4,7,18,22–26,29,44]). In particular, Gromoll and Meyer [20] of 1969 established the existence of infinitely many distinct closed geodesics on M, provided that the Betti number sequence $\{b_p(\Lambda M; \mathbf{Q})\}_{p\in\mathbf{N}}$ of the free loop space ΛM of M is unbounded. Then in [39] of 1976, for compact simply connected manifold M, Vigué-Poirrier and Sullivan further proved this Betti number sequence is bounded if and only if M satisfies

$$H^*(M; \mathbf{Q}) \cong T_{d,n+1}(x) = \mathbf{Q}[x]/(x^{n+1} = 0)$$

with a generator x of degree $d \geq 2$ and height $n + 1 \geq 2$, where $\dim M = dn$. Due to their works, many mathematicians are concerned with the compact globally symmetric spaces of rank 1 which consist in

$$S^n, \quad \mathbb{R}P^n, \quad \mathbb{C}P^n, \quad \mathbb{H}P^n \quad and \quad CaP^2,$$

where the Gromoll–Meyer assumption does not hold.

Recently the Maslov-type index theory for symplectic paths has been applied to study the closed geodesic problem on simply connected manifolds. In 2005, Bangert and Long proved the existence of at least two distinct closed geodesics on every Finsler (S^2, F) (which was published as [6] in 2010). Since then in the last ten years, a great number of results on the multiplicity of closed geodesics on simply connected Finsler manifolds have appeared, for which we refer readers to [9,10,27,33–35,40,41,37,21,12,13] and the references therein.

As for the multiplicity of closed geodesics on non-simply connected manifolds whose free loop space possesses bounded Betti number sequence, Ballman et al. [2] proved in 1981 that every Riemannian manifold with the fundamental group being a nontrivial finitely cyclic group and possessing a generic metric has infinitely many distinct closed geodesics. In 1984, Bangert and Hingston [5] proved that any Riemannian manifold with the fundamental group being an infinite cyclic group has infinitely many distinct closed geodesics. Since then, there seem to be very few works on the multiplicity of closed geodesics on non-simply connected manifolds. The main reason is that the topological structures of the free loop spaces on these manifolds are not well known, so that the classical Morse theory is difficult to be applicable. In [43] of