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Existence and mass concentration of 2D attractive Bose–Einstein condensates with periodic potentials ☆

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Abstract

In this paper we consider a two-dimensional attractive Bose–Einstein condensate with periodic potential, described by Gross–Pitaevskii (GP) functional. By concentration-compactness lemma we show that minimizers of this functional exist when the interaction strength a satisfies $a_* < a < a^*$ for some constants $a_* \geq 0$, $a^* > 0$, and there is no minimizer for $a \geq a^*$. When a approaches a^*, using concentration-compactness arguments again we obtain an optimal energy estimate depending on the shape of periodic potential. Moreover, we analyze the mass concentration.

Keywords: Bose–Einstein condensate; Attractive interaction; GP functional; Periodic potential; Energy estimate; Mass concentration

1. Introduction and main results

The remarkable experiments [1,5] of cold atom physics during the last two decades gave a new impetus to the theory of Bose–Einstein condensates. In such a state, many interacting bosons occupy the same quantum state which is often called ground state, and may thus be described by Gross–Pitaevskii (GP) [12,25] functional. Several rigorous mathematical verifications of GP

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theory were established [19–22]. In this paper, we will study the following two-dimensional (2D) GP energy functional with attractive interaction and periodic potential

$$E_a(u) := \int_{\mathbb{R}^2} (|\nabla u(x)|^2 + V(x)|u(x)|^2)\,dx - \frac{a}{2}\int_{\mathbb{R}^2} |u|^4\,dx, \quad u \in H^1(\mathbb{R}^2), \tag{1.1}$$

where $a > 0$, describes the strength of the attractive interactions, and $V(x)$ is some external periodic potential. The ground state of the system is obtained by minimizing (1.1) under the unit mass constraint

$$\int_{\mathbb{R}^2} |u(x)|^2\,dx = 1. \tag{1.2}$$

Alternatively, one may want to impose the constraint $\int_{\mathbb{R}^2} |u(x)|^2\,dx = N$, where N is the particle number, but this latter case can easily be reduced to the previous one if replacing a by a/N. Hence we can only consider (1.2) and define the GP energy to be

$$e(a) := \inf_{u\in H^1(\mathbb{R}^2),\ \|u\|_2^2=1} E_a(u). \tag{1.3}$$

It is known that Bose–Einstein condensates with attractive interactions have an interesting phenomenon, that is, the condensation system would collapse if the particle number increases beyond a threshold value, see [2,3,5,17,26]. It is interesting to investigate details of this collapse mathematically. When the potential $V(x)$ is trapped, i.e. $\lim_{|x|\to\infty} V(x) = \infty$, recently there is some corresponding collapse analysis, see [7,13,14]. Guo and Seiringer [13] proved that there exists a critical number $a^* > 0$, such that the minimizers of 2D GP energy functional (1.1) exist for $0 < a < a^*$ and no minimizer exists for $a \ge a^*$. They also showed that $a^* = \|Q\|_2^2$, where Q is the unique positive radial solution of nonlinear scalar field equation

$$-\Delta u + u - u^3 = 0 \ \text{ in } \mathbb{R}^2, \text{ where } u \in H^1(\mathbb{R}^2). \tag{1.4}$$

Moreover, a detailed analysis of the minimizer as a approaches a^* was given in [13], in which they showed that all the mass concentrates at a global minimum of the trapping potential. Soon afterwards Ring-shaped trapping potentials [14] and inhomogeneous attractive interactions with trapping potentials [7] were also considered.

However, when $V(x)$ is a periodic potential, to our knowledge, there is no result of collapse analysis of minimization problem (1.3). Periodic potential also plays an important role in experiments and theory of Bose–Einstein condensates [6,8,9,18,24,27]. Periodic potentials may be induced by optical lattices [24], written into optical media as permanent structures [27], or photoinduced as virtual lattices by interfering pump beams in photorefractive crystals [18]. In this paper we make the following assumptions on the periodic potential $V(x)$.

(V₁) $V(x) \in C(\mathbb{R}^2)$, is an 1-period function, $V(x+z) = V(x)$ for all $x \in \mathbb{R}^2$ and $z \in \mathbb{Z}^2$;
(V₂) $\Lambda > \min_{\mathbb{R}^2} V(x)$, where $\Lambda := \inf\{\int_{\mathbb{R}^2} |\nabla u|^2 + V(x)|u|^2\,dx : u \in H^1(\mathbb{R}^2),\ \|u\|_2^2 = 1\}$.

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