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On the heat diffusion starting with degeneracy

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Abstract

In this article we study the instant smoothing property of the heat diffusion that starts with degeneracy:

$$u_t(t,x) = t^{\alpha}\Delta u + f(t,x), \quad t \in (0,T),\ x \in \mathbb{R}^d \ ;\ u(0,x) = u_0(x),$$

where $\alpha \in (-1,\infty)$. We provide the existence and uniqueness result in an appropriate Sobolev space setting. For a fixed f the regularity improvement in Sobolev regularity from u_0 to u changes continuously along α. In particular, the larger $\alpha > 0$, the smaller the improvement is. Moreover, we study a regularity relation between f and u near time $t = 0$ as α varies.

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1. Introduction

A usual model for heat type diffusion in the whole space is

$$u_t(t,x) = Lu(t,x) + f(t,x), \quad t \in (0,T),\ x \in \mathbb{R}^d \ ;\ u(0,x) = u_0(x), \tag{1.1}$$

where the operator L is defined by $Lu(t,x) = \sum_{i,j=1}^d a^{ij}(t)u_{x^i x^j}(t,x)$ with the strong parabolicity condition of the diffusion coefficients a^{ij}. Namely, there is a strictly positive constant δ such that for all $t > 0$ and any nonzero vector $\xi = (\xi^1, \ldots, \xi^d) \in \mathbb{R}^d$

$$\sum_{i,j=1}^d a^{ij}(t)\xi_i\xi_j \geq \delta|\xi|^2 > 0$$

holds. One of the main consequences under the strong parabolicity condition is the instant smoothing property of the diffusion. In the Sobolev space setting, the regularity relation between the initial condition u_0 and the solution u is given by, for instance,

$$u_0 \in B_p^{2-\frac{2}{p}} \quad \Rightarrow \quad u \in L_p((0,T), W_p^2) \tag{1.2}$$

provided that f belongs to $L_p((0,T), L_p)$, where $p > 1$ and $B_p^{2-2/p} = B_p^{2-2/p}(\mathbb{R}^d)$ is a Besov space. The number 2 in the fraction $2/p$ also appears when we have Laplace operator Δ instead of the general operator L. In fact, the role of the strong parabolicity condition in view of instant smoothing effect is to make the solution of the problem (1.1) behave in the same manner as the case with $L = \Delta$ near $t = 0$.

In this article we address the instant smoothing effect of the heat diffusion which starts with degeneracy at time 0 and active for $t > 0$. Our basic model problem is

$$u_t(t,x) = t^\alpha \Delta u + f(t,x), \quad t \in (0,T),\ x \in \mathbb{R}^d \ ;\ u(0,x) = u_0(x), \tag{1.3}$$

where $\alpha \in (-1, \infty)$. We prove uniqueness and existence of a solution of the problem (1.3) in an appropriate space and present two kinds of instant smoothing properties

$$\alpha > \frac{1}{p} - 1,\ u_0 \in B_p^{2-\frac{2}{p(\alpha+1)}}, \quad f \equiv 0 \quad \Rightarrow \quad u \in L_p((0,T), W_p^2) \tag{1.4}$$

and

$$\alpha > -1,\ u_0 \in B_p^{\frac{2}{\alpha+1} - \frac{2}{p(\alpha+1)}} \quad \Rightarrow \quad u,\ t^{\alpha/2}u_{x^i},\ t^\alpha u_{x^i x^j} \in L_p((0,T), L_p) \tag{1.5}$$

provided $f \in L_p((0,T), L_p)$. Note that if $\alpha = 0$ then above results coincide with the classical one presented in (1.2). We remark that from both (1.4) and (1.5) we can observe that the activation of the heat diffusion and hence the instant smoothing effect get slower as α gets larger.

In the literature there are numerous articles handling various type of degenerate equations. One can find some classical results for instance in [2,3,8–10,7]. The typical degeneracy in the literature is either too general (e.g. $(a^{ij}(t,x)) \geq 0$) or depends only on the space variable, for