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Asymptotic behavior of gradient-like dynamical systems involving inertia and multiscale aspects

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Abstract

In a Hilbert space $\mathbb{H}$, we study the asymptotic behavior, as time variable t goes to $+\infty$, of nonautonomous gradient-like dynamical systems involving inertia and multiscale features. Given $\Phi : \mathbb{H} \to \mathbb{R}$ and $\Psi : \mathbb{H} \to \mathbb{R}$ two convex differentiable functions, γ a positive damping parameter, and $\epsilon(t)$ a function of t which tends to zero as t goes to $+\infty$, we consider the second-order differential equation

$$\ddot{x}(t) + \gamma \dot{x}(t) + \nabla\Phi(x(t)) + \epsilon(t)\nabla\Psi(x(t)) = 0.$$

This system models the emergence of various collective behaviors in game theory, as well as the asymptotic control of coupled nonlinear oscillators. Assuming that $\epsilon(t)$ tends to zero moderately slowly as t goes to infinity, we show that the trajectories converge weakly in $\mathbb{H}$. The limiting equilibria are solutions of the hierarchical minimization problem which consists in minimizing Ψ over the set C of minimizers of Φ. As key assumptions, we suppose that $\int_0^{+\infty} \epsilon(t)dt = +\infty$ and that, for every p belonging to a convex cone $\mathcal{C}$ depending on the data Φ and Ψ

$$\int_0^{+\infty} \left[\Phi^*\left(\epsilon(t)p\right) - \sigma_C\left(\epsilon(t)p\right)\right] dt < +\infty$$

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where Φ^* is the Fenchel conjugate of Φ, and σ_C is the support function of C. An application is given to coupled oscillators.

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1. Introduction

$\mathbb{H}$ is a real Hilbert space, we write $\|x\|^2 = \langle x, x\rangle$ for $x \in \mathbb{H}$. For any differentiable function $F : \mathbb{H} \to \mathbb{R}$, its gradient is denoted by ∇F. Thus $F'(x)(y) = \langle \nabla F(x), y\rangle$. The first order (respectively second order) derivative at time t of a function $x(\cdot) : [0, +\infty[\to \mathbb{H}$ is denoted by $\dot{x}(t)$ (respectively $\ddot{x}(t)$). Throughout the paper γ is a fixed positive parameter (viscous damping coefficient).

1.1. Problem statement

Henceforth, we make the following standing assumptions $(\mathcal{H}_0)$ on data Φ, Ψ and $\epsilon(\cdot)$, that will be needed throughout the paper. We write $S = \operatorname{argmin}_C \Psi$ with $C = \operatorname{argmin} \Phi$, or, equivalently, $S = \operatorname{argmin}\{\Psi|\operatorname{argmin}\Phi\}$.

$$\begin{cases} \Phi : \mathbb{H} \to \mathbb{R} \quad \text{is convex, } \nabla\Phi \text{ is Lipschitz continuous on bounded sets of } \mathbb{H}, \\ \qquad\qquad C = \operatorname{argmin}\Phi = \Phi^{-1}(0) \neq \emptyset; \\ \Psi : \mathbb{H} \to \mathbb{R} \text{ is convex, } \nabla\Psi \text{ is Lipschitz continuous on bounded sets of } \mathbb{H}, \\ \qquad\qquad \Psi \text{ is bounded from below;} \\ S = \operatorname{argmin}\{\Psi|\operatorname{argmin}\Phi\} \neq \emptyset; \\ \epsilon(\cdot) : [0, +\infty[\to]0, +\infty[\text{ is a nonincreasing function, of class } C^1, \\ \qquad\qquad \epsilon(t) \text{ and } \dot{\epsilon}(t) \text{ tend to zero as } t \to +\infty. \end{cases} \tag{$\mathcal{H}_0$}$$

We study the asymptotic behavior ($t \to +\infty$) of the trajectories of the nonautonomous Multiscaled Inertial Gradient-like system ((MIG) for short)

$$\ddot{x}(t) + \gamma\dot{x}(t) + \nabla\Phi(x(t)) + \epsilon(t)\nabla\Psi(x(t)) = 0. \tag{MIG}$$

We want to give sufficient conditions on $\epsilon(\cdot)$ (typically slow decay to zero as $t \to +\infty$) ensuring that each trajectory of (MIG) asymptotically converges (as $t \to +\infty$) to a minimizer of Φ, which also minimizes Ψ over all minima of Φ. It is an asymptotic hierarchical minimization property.

Let us observe that when $\Psi \equiv 0$, or $\epsilon = 0$, the above dynamic reduces to

$$\ddot{x}(t) + \gamma\dot{x}(t) + \nabla\Phi(x(t)) = 0. \tag{HBF}$$

The Heavy Ball with Friction dynamical system, (HBF) for short, plays an important role in mechanics, control theory, and optimization, see [1,5,10,18,20] for a general presentation. Because